Supplementary Material for "Higher-Order Model Checking of Effect-Handling Programs with Answer-Type Modification"

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August 26, 2024

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1 Outline

This is the supplementary material of the paper titled "Higher-Order Model Checking of Effect-Handling Programs with Answer-Type Modification" published at OOPSLA'24, including all the definitions, lemmas, theorems, and proofs mentioned in the paper.

2 Definition

2.1 Trees

Definition 1 (Tree Constructor Signatures)**.** *A tree constructor signature S is a map from tree constructors, ranged over by s, to natural numbers that represent the arities of the constructors. We write $ar_S(s)$ for the arity of s assigned by S.*

Definition 2 (Finitely Branching Infinite Trees)**.** *The set $\mathbf{Tree}_S$ of finitely branching (possibly) infinite trees generated by a tree constructor signature S is defined coinductively by the following grammar (where s is in the domain of S):*

$$t \quad ::= \quad \bot \mid s(t_1, \cdots, t_{ar_S(s)}) \; .$$

Evaluation rules $\boxed{M_1 \longrightarrow M_2}$

$$
\begin{array}{rcll}
(\lambda x.M_1)\ V_2 & \longrightarrow & M_1[V_2/x] & & \text{HE\_Beta} \\
(\mathsf{fix}\,x.\,V_1)\ V_2 & \longrightarrow & V_1[\mathsf{fix}\,x.\,V_1/x]\ V_2 & & \text{HE\_Fix} \\
\mathsf{case}(\underline{i};M_1,\cdots,M_n) & \longrightarrow & M_i & (\text{if } 0 < i \le n) & \text{HE\_Case} \\
\mathsf{let}\,x = \mathsf{return}\ V_1\ \mathsf{in}\ M_2 & \longrightarrow & M_2[V_1/x] & & \text{HE\_LetV} \\
\mathsf{let}\,x = \sigma(V_1;y.\,M_1)\ \mathsf{in}\ M_2 & \longrightarrow & \sigma(V_1;y.\,\mathsf{let}\,x = M_1\ \mathsf{in}\ M_2) & (\text{if } y \notin \mathit{fv}(M_2)) & \text{HE\_LetOp} \\
\mathsf{with}\ H\ \mathsf{handle}\ \mathsf{return}\ V & \longrightarrow & M[V/x] & (\text{if } \mathsf{return}\,x \mapsto M \in H) & \text{HE\_HandleV} \\
\mathsf{with}\ H\ \mathsf{handle}\ \sigma(V;y.\,M) & \longrightarrow & M'[V/x][\lambda y.\mathsf{with}\ H\ \mathsf{handle}\ M/k] & (\text{if } \sigma(x;k) \mapsto M' \in H) & \text{HE\_HandleOp}
\end{array}
$$

$$
\frac{M_1 \longrightarrow M_1'}{\mathsf{let}\,x = M_1\ \mathsf{in}\ M_2 \longrightarrow \mathsf{let}\,x = M_1'\ \mathsf{in}\ M_2}\ \text{HE\_LetE}
\qquad
\frac{M \longrightarrow M'}{\mathsf{with}\ H\ \mathsf{handle}\ M \longrightarrow \mathsf{with}\ H\ \mathsf{handle}\ M'}\ \text{HE\_HandleE}
$$

Figure 1: Semantics.

2.2 HEPCF$^{\mathsf{ATM}}$: PCF with Answer-Type Modification for Algebraic Effects and Handlers

2.2.1 Syntax

Variables x, y, z, f, h, k **Operations** σ

$$
\begin{array}{rrcl}
\textbf{Base types} & B & ::= & \mathsf{bool} \mid \mathsf{unit} \mid \cdots \\
\textbf{Enum types} & E & ::= & 1 \mid 2 \mid \cdots \\
\textbf{Value types} & T & ::= & B \mid E \mid T \to C \\
\textbf{Computation types} & C & ::= & \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}} \\
\textbf{Operation signatures} & \Sigma & ::= & \{\sigma_i : T_i^{\mathrm{par}} \rightsquigarrow T_i^{\mathrm{ari}} \,/\, A_i^{\mathrm{ini}} \Rightarrow A_i^{\mathrm{fin}}\}^{1\le i\le n} \\
\textbf{Answer types} & A & ::= & T \mid C \\
\textbf{Base constants} & c & ::= & \mathsf{true} \mid \mathsf{false} \mid () \mid \cdots \\
\textbf{Enum constants} & \varepsilon & ::= & \underline{1} \mid \underline{2} \mid \cdots \\
\textbf{Values} & V & ::= & x \mid c \mid \varepsilon \mid \lambda x.M \mid \mathsf{fix}\,x.\,V \\
\textbf{Terms} & M & ::= & \mathsf{return}\ V \mid \mathsf{let}\,x = M_1\ \mathsf{in}\ M_2 \mid V_1\ V_2 \mid \mathsf{case}(V;M_1,\cdots,M_n) \mid \\
 & & & \sigma(V;x.\,M) \mid \mathsf{with}\ H\ \mathsf{handle}\ M \\
\textbf{Handlers} & H & ::= & \{\mathsf{return}\,x \mapsto M\} \uplus \{\sigma_i(x_i;k_i) \mapsto M_i\}^{1\le i\le n} \\
\textbf{Typing contexts} & \Gamma & ::= & \emptyset \mid \Gamma, x : T
\end{array}
$$

Convention 1. *We write Γ_1, Γ_2 for the concatenation of Γ_1 and Γ_2. For a computation type $C = \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}$, we write $C.\Sigma$ for the operation signature Σ.*

Definition 3 (Free variables and substitution)**.** *The set $\mathit{fv}(M)$ of free variables in a term M is defined in a standard manner. Value substitution $M[V/x]$ and $V'[V/x]$ of V for x in M and V', respectively, are defined in a capture-avoiding manner as usual.*

Assumption 1. *We assume a function ty that assigns a base type to every constant c.*

Definition 4 (Top-Level Operation Signatures)**.** *An operation signature Σ is* top-level *if, for any $\sigma : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}} \in \Sigma$, $T^{\mathrm{par}} = B$ for some B, $T^{\mathrm{ari}} = E$ for some E, and $A^{\mathrm{ini}} = A^{\mathrm{fin}} = T$ for some T.*

Definition 5 (Ground Types)**.** *A type T is ground if and only if $T = B$ for some B or $T = E$ for some E.*

2.2.2 Semantics

Definition 6 (Semantics)**.** *The evaluation relation $M_1 \longrightarrow M_2$ is the smallest relations satisfying the rules in Figure 1.*

Definition 7 (Multi-step evaluation)**.** *We write $M \longrightarrow^n M'$ if and only if there exist some terms $M_0, \cdots, M_n$ such that: $M = M_0$; $\forall\, i < n.\ M_i \longrightarrow M_{i+1}$; and $M_n = M'$. We write $M \longrightarrow^* M'$ if and only if $M \longrightarrow^n M'$ for some n.*

Typing rules $\boxed{\Gamma \vdash V : T}$ $\boxed{\Gamma \vdash M : C}$

$$\frac{}{\Gamma \vdash x : \Gamma(x)}\ \textsc{HT\_Var} \qquad \frac{}{\Gamma \vdash c : ty(c)}\ \textsc{HT\_Const} \qquad \frac{0 < i \leq n}{\Gamma \vdash \underline{\mathsf{i}} : \mathsf{n}}\ \textsc{HT\_EConst}$$

$$\frac{\Gamma, x : T \vdash M : C}{\Gamma \vdash \lambda x.M : T \to C}\ \textsc{HT\_Abs} \qquad \frac{\Gamma, x : T \to C \vdash V : T \to C}{\Gamma \vdash \mathsf{fix}\, x.V : T \to C}\ \textsc{HT\_Fix}$$

$$\frac{\Gamma \vdash V : T}{\Gamma \vdash \mathsf{return}\ V : \Sigma \triangleright T \,/\, A \Rightarrow A}\ \textsc{HT\_Return}$$

$$\frac{\Gamma \vdash M_1 : \Sigma \triangleright T_1 \,/\, A \Rightarrow A_1 \quad \Gamma, x : T_1 \vdash M_2 : \Sigma \triangleright T_2 \,/\, A_2 \Rightarrow A}{\Gamma \vdash \mathsf{let}\, x = M_1\, \mathsf{in}\, M_2 : \Sigma \triangleright T_2 \,/\, A_2 \Rightarrow A_1}\ \textsc{HT\_Let}$$

$$\frac{\Gamma \vdash V_1 : T \to C \quad \Gamma \vdash V_2 : T}{\Gamma \vdash V_1\ V_2 : C}\ \textsc{HT\_App} \qquad \frac{\Gamma \vdash V : \mathsf{n} \quad \forall\, i \in [1, n].\ \Gamma \vdash M_i : C}{\Gamma \vdash \mathsf{case}(V; M_1, \cdots, M_n) : C}\ \textsc{HT\_Case}$$

$$\frac{\Sigma \ni \sigma : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}} \quad \Gamma \vdash V : T^{\mathrm{par}} \quad \Gamma, x : T^{\mathrm{ari}} \vdash M : \Sigma \triangleright T \,/\, A \Rightarrow A^{\mathrm{ini}}}{\Gamma \vdash \sigma(V; x.\, M) : \Sigma \triangleright T \,/\, A \Rightarrow A^{\mathrm{fin}}}\ \textsc{HT\_Op}$$

$$\frac{\begin{array}{c} H = \{\mathsf{return}\, x \mapsto M'\} \uplus \{\sigma_i(x_i; k_i) \mapsto M_i\}^{1 \leq i \leq n} \quad \Sigma = \{\sigma_i : T_i^{\mathrm{par}} \rightsquigarrow T_i^{\mathrm{ari}} \,/\, C_i^{\mathrm{ini}} \Rightarrow C_i^{\mathrm{fin}}\}^{1 \leq i \leq n} \\ \Gamma \vdash M : \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}} \quad \Gamma, x : T \vdash M' : C^{\mathrm{ini}} \quad \forall\, i \in [1, n].\ \Gamma, x_i : T_i^{\mathrm{par}}, k_i : T_i^{\mathrm{ari}} \to C_i^{\mathrm{ini}} \vdash M_i : C_i^{\mathrm{fin}} \end{array}}{\Gamma \vdash \mathsf{with}\ H\ \mathsf{handle}\ M : C^{\mathrm{fin}}}\ \textsc{HT\_Handle}$$

Figure 2: Type system.

Definition 8 (Infinite Evaluation)**.** *We write $M \longrightarrow^{\omega}$ if and only if, for any natural number n, there exists some term M' such that $M \longrightarrow^{n} M'$.*

Definition 9 (Nonreducible terms)**.** *We write $M \not\longrightarrow$ if and only if there is no M' such that $M \longrightarrow M'$.*

2.2.3 Type System

Definition 10 (Domains of Typing Contexts)**.** *Given a typing context Γ, its domain $dom(\Gamma)$ is defined by induction on Γ as follows.*

$$\begin{array}{lcl} dom(\emptyset) & \stackrel{\mathrm{def}}{=} & \emptyset \\ dom(\Gamma, x : T) & \stackrel{\mathrm{def}}{=} & \{x\} \cup dom(\Gamma) \end{array}$$

Definition 11 (Typing Contexts as Functions)**.** *We view Γ as a function that maps a variable to a type. $\Gamma(x) = T$ if and only if $x : T \in \Gamma$.*

Definition 12 (Typing)**.** *The typing of values (with judgments of the form $\Gamma \vdash V : T$) and terms (with judgments of the form $\Gamma \vdash M : C$) is the smallest relation satisfying the rules in Figure 2.*

2.2.4 Effect Trees

Definition 13 (Effect Trees for $\mathsf{HEPCF}^{\mathsf{ATM}}$ Computations)**.** *Given an operation signature Σ and a type T, the tree constructor signature S_T^{Σ} is defined as follows:*

$$S_T^{\Sigma} \stackrel{\mathrm{def}}{=} \{\sigma : n+1 \mid \sigma : B \rightsquigarrow \mathsf{n} \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}} \in \Sigma\} \cup \{\mathsf{return}\ V : 0 \mid \emptyset \vdash V : T\} \cup \bigcup_{c} \{c : 0\}\ .$$

where, for a tree constructor s (that is an operation σ, return construct return V*, or base constant c), $s : n$ denotes the pair (s, n), meaning that the arity of s is n. Given a term M such that $\emptyset \vdash M : \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}$, the effect tree of M, denoted by $\mathbf{ET}(M)$, is a tree in $\mathbf{Tree}_{S_T^{\Sigma}}$ defined by the following (possibly infinite) process:*

- *if $M \longrightarrow^{\omega}$, then $\mathbf{ET}(M) = \bot$;*

- *if* $M \longrightarrow^* \mathsf{return}\ V$, *then* $\mathbf{ET}(M) = \mathsf{return}\ V$; *and*
- *if* $M \longrightarrow^* \sigma(c; x.\, M')$ *and* $\sigma : B \rightsquigarrow \mathsf{n}\ /\ A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}} \in \Sigma$, *then* $\mathbf{ET}(M) = \sigma(c, \mathbf{ET}(M'[\underline{\mathsf{1}}/x]), \cdots, \mathbf{ET}(M'[\underline{\mathsf{n}}/x]))$.

Evaluation rules $\boxed{e_1 \longrightarrow e_2}$

$$\begin{array}{rcll}
(\lambda x. e_1)\, v_2 & \longrightarrow & e_1[v_2/x] & & \text{E_Beta} \\
(\mathsf{fix}\, x. v_1)\, v_2 & \longrightarrow & v_1[\mathsf{fix}\, x. v_1/x]\, v_2 & & \text{E_Fix} \\
\mathsf{case}(\underline{i}; e_1, \cdots, e_n) & \longrightarrow & e_i & (\text{if } 0 < i \leq n) & \text{E_Case} \\
\mathsf{let}\, x = \mathsf{return}\, v_1\, \mathsf{in}\, e_2 & \longrightarrow & e_2[v_1/x] & & \text{E_LetV} \\
\mathsf{let}\, x = \sigma(v_1; y.\, e_1)\, \mathsf{in}\, e_2 & \longrightarrow & \sigma(v_1; y.\, \mathsf{let}\, x = e_1\, \mathsf{in}\, e_2) & (\text{if } y \notin fv(e_2)) & \text{E_LetOp}
\end{array}$$

$$\frac{e_1 \longrightarrow e_1'}{\mathsf{let}\, x = e_1\, \mathsf{in}\, e_2 \longrightarrow \mathsf{let}\, x = e_1'\, \mathsf{in}\, e_2}\ (\text{E\_LetE})$$

Figure 3: Semantics.

2.3 EPCF: PCF with Algebraic Effects

2.3.1 Syntax

Variables x, y, z, f, h, k **Operations** σ

$$\begin{array}{rrcl}
\textbf{Base types} & B & ::= & \mathsf{bool} \mid \mathsf{unit} \mid \cdots \\
\textbf{Enum types} & E & ::= & \mathsf{1} \mid \mathsf{2} \mid \cdots \\
\textbf{Types} & \tau & ::= & B \mid E \mid \tau_1 \to \tau_2 \\
\textbf{Operation signatures} & \Xi & ::= & \{\sigma_i : B_i \rightsquigarrow E_i\}^{1 \leq i \leq n} \\
\textbf{Base constants} & c & ::= & \mathsf{true} \mid \mathsf{false} \mid () \mid \cdots \\
\textbf{Enum constants} & \varepsilon & ::= & \underline{1} \mid \underline{2} \mid \cdots \\
\textbf{Values} & v & ::= & x \mid c \mid \varepsilon \mid \lambda x. e \mid \mathsf{fix}\, x. v \\
\textbf{Terms} & e & ::= & \mathsf{return}\, v \mid \mathsf{let}\, x = e_1\, \mathsf{in}\, e_2 \mid v_1\, v_2 \mid \mathsf{case}(v; e_1, \cdots, e_n) \mid \sigma(v; x.\, e) \\
\textbf{Typing contexts} & \Delta & ::= & \emptyset \mid \Delta, x : \tau
\end{array}$$

For the syntactic operations common in HEPCF$^{\mathsf{ATM}}$ and EPCF, we use the same notation (e.g., $fv(e)$ is the set of free variables in e and $e[v/x]$ is the term obtained by substituting v for x in e).

2.3.2 Semantics

Definition 14 (Semantics). *The evaluation relation $e_1 \longrightarrow e_2$ is the smallest relations satisfying the rules in Figure 3.*

Definition 15 (Multi-step evaluation). *We write $e \longrightarrow^n e'$ if and only if there exist some terms $e_0, \cdots, e_n$ such that: $e = e_0$; $\forall\, i < n.\ e_i \longrightarrow e_{i+1}$; and $e_n = e'$. We write $e \longrightarrow^* e'$ if and only if $e \longrightarrow^n e'$ for some n, and $e \longrightarrow^+ e'$ if and only if $e \longrightarrow^n e'$ for some $n > 0$.*

Definition 16 (Infinite Evaluation). *We write $e \longrightarrow^\omega$ if and only if, for any natural number n, there exists some term e' such that $e \longrightarrow^n e'$.*

Definition 17 (Nonreducible terms). *We write $e \not\longrightarrow$ if and only if there is no e' such that $e \longrightarrow e'$.*

2.3.3 Type System

Definition 18 (Typing). *Fix an operation signature Ξ. Then, the typing of values (with judgments of the form $\Xi \parallel \Delta \vdash v : \tau$) and terms (with judgments of the form $\Xi \parallel \Delta \vdash e : \tau$) is the smallest relation satisfying the rules in Figure 4.*

2.3.4 Effect Trees

Definition 19 (Effect Trees for EPCF Computations). *Given an operation signature Ξ and a type τ, the tree constructor signature S^{Ξ}_{τ} is defined as follows:*

$$S^{\Xi}_{\tau} \stackrel{\text{def}}{=} \{\sigma : n+1 \mid \sigma : B \rightsquigarrow \mathsf{n} \in \Xi\} \cup \{\mathsf{return}\, v : 0 \mid \Xi \parallel \emptyset \vdash v : \tau\} \cup \bigcup_{c} \{c : 0\}\ .$$

Typing rules $\boxed{\Xi \parallel \Delta \vdash v : \tau}$ $\boxed{\Xi \parallel \Delta \vdash e : \tau}$

$$\frac{}{\Xi \parallel \Delta \vdash x : \Delta(x)}\ \text{T\_VAR} \qquad \frac{}{\Xi \parallel \Delta \vdash c : ty(c)}\ \text{T\_CONST} \qquad \frac{0 < i \leq n}{\Xi \parallel \Delta \vdash \underline{\mathsf{i}} : \mathsf{n}}\ \text{T\_ECONST}$$

$$\frac{\Xi \parallel \Delta, x : \tau_1 \vdash e : \tau_2}{\Xi \parallel \Delta \vdash \lambda x.e : \tau_1 \to \tau_2}\ \text{T\_ABS} \qquad \frac{\Xi \parallel \Delta, x : \tau_1 \to \tau_2 \vdash v : \tau_1 \to \tau_2}{\Xi \parallel \Delta \vdash \mathsf{fix}\, x.v : \tau_1 \to \tau_2}\ \text{T\_FIX}$$

$$\frac{\Xi \parallel \Delta \vdash v : \tau}{\Xi \parallel \Delta \vdash \mathsf{return}\, v : \tau}\ \text{T\_RETURN} \qquad \frac{\Xi \parallel \Delta \vdash e_1 : \tau_1 \quad \Xi \parallel \Delta, x : \tau_1 \vdash e_2 : \tau_2}{\Xi \parallel \Delta \vdash \mathsf{let}\, x = e_1\, \mathsf{in}\, e_2 : \tau_2}\ \text{T\_LET}$$

$$\frac{\Xi \parallel \Delta \vdash v_1 : \tau_1 \to \tau_2 \quad \Xi \parallel \Delta \vdash v_2 : \tau_1}{\Xi \parallel \Delta \vdash v_1\, v_2 : \tau_2}\ \text{T\_APP} \qquad \frac{\Xi \parallel \Delta \vdash v : \mathsf{n} \quad \forall i \in [1, n].\ \Xi \parallel \Delta \vdash e_i : \tau}{\Xi \parallel \Delta \vdash \mathsf{case}(v; e_1, \cdots, e_n) : \tau}\ \text{T\_CASE}$$

$$\frac{\Xi \ni \sigma : B \rightsquigarrow E \quad \Xi \parallel \Delta \vdash v : B \quad \Xi \parallel \Delta, x : E \vdash e : \tau}{\Xi \parallel \Delta \vdash \sigma(v; x.\, e) : \tau}\ \text{T\_OP}$$

Figure 4: Type system.

Given a term e such that $\Xi \parallel \emptyset \vdash e : \tau$, the effect tree of e, denoted by $\mathbf{ET}(e)$, is a tree in $\mathbf{Tree}_{S^{\Xi}_{\tau}}$ defined by the following (possibly infinite) process:

- *if $e \longrightarrow^{\omega}$, then $\mathbf{ET}(e) = \bot$;*
- *if $e \longrightarrow^{*} \mathsf{return}\, v$, then $\mathbf{ET}(e) = \mathsf{return}\, v$; and*
- *if $e \longrightarrow^{*} \sigma(c; x.\, e')$ and $\sigma : B \rightsquigarrow \mathsf{n} \in \Xi$, then $\mathbf{ET}(e) = \sigma(c, \mathbf{ET}(e'[\underline{1}/x]), \cdots, \mathbf{ET}(e'[\underline{\mathsf{n}}/x]))$.*

2.4 CPS Transformation from $\mathsf{HEPCF}^{\mathsf{ATM}}$ to EPCF

Our CPS transformation is defined using the following shorthand:

- A sequence of entities $a_1, \cdots, a_n$ is abbreviated to $\overline{a}$, and its length is denoted by $|\overline{a}|$. Given $\overline{a}$, we write a_i to designate the i-th element of the sequence $\overline{a}$.

- Given a variable sequence $\overline{x} = x_1, \cdots, x_n$, we write $\lambda\overline{x}.e$ for the EPCF term $\lambda x_1.\mathsf{return}\,\lambda x_2.\cdots\,\mathsf{return}\,\lambda x_n.e$.

- Given a term e and values $v_1, \cdots, v_n$ $(n > 0)$, we write $e\,v_1\,\cdots\,v_n$ for the EPCF term $\mathsf{let}\,x_0 = e\,\mathsf{in}\,\mathsf{let}\,x_1 = x_0\,v_1\,\mathsf{in}\,\mathsf{let}\,x_2 = x_1\,v_2\,\mathsf{in}\,\cdots\,\mathsf{let}\,x_{n-1} = x_{n-2}\,v_{n-1}\,\mathsf{in}\,x_{n-1}\,v_n$ where the variables $x_0, x_1, \cdots x_{n-1}$ are assumed to be fresh.

We also assume that the set of all the operations is totally ordered.

Definition 20 (CPS Transformation of Types, Values, and Terms)**.** *CPS Transformation* $[\![-]\!]$ *from* $\mathsf{HEPCF}^{\mathsf{ATM}}$ *to* EPCF *is defined in Figure 5, mapping*

- *value types* T *to* EPCF *types* $[\![T]\!]$,
- *computation types* C *to* EPCF *types* $[\![C]\!]$,
- *operation signatures* Σ *to functions that, given a* EPCF *type* τ*, return the* EPCF *type* $[\![\Sigma]\!][\,\tau\,]$,
- *values* V *to* EPCF *values* $[\![V]\!]$,
- *terms* M *to* EPCF *values* $[\![M]\!]$*, and*
- *terms* M *to* EPCF *terms* $[\![M]\!][\,\overline{v^{\mathsf{h}}}\mid v^{\mathsf{k}}\,]$ *given values* $\overline{v^{\mathsf{h}}}$ *and* v^{k}.

The definition of $[\![M]\!]$ *and* $[\![M]\!][\,\overline{v^{\mathsf{h}}}\mid v^{\mathsf{k}}\,]$ *assumes that the* $\mathsf{HEPCF}^{\mathsf{ATM}}$ *term* M *to be CPS-transformed is well typed. In general, given an* $\mathsf{HEPCF}^{\mathsf{ATM}}$ *term* M *typed at a computation type with an operation signature* Σ*, the CPS-transformation result* $[\![M]\!]$ *takes the form* $\lambda\overline{h},k.e$ *for some variables* $\overline{h},k$ *and* EPCF *term* e *such that* $|\overline{h}| = |\Sigma|$*. Similarly,* $[\![M]\!][\,\overline{v^{\mathsf{h}}}\mid v^{\mathsf{k}}\,]$ *assumes that* $|\Sigma|$ *values* $\overline{v^{\mathsf{h}}}$ *are given. The transformation of operation calls assumes that a called operation* σ_i *is the* i*-th operation in* Σ *(under the order of operations). We also write* $[\![\Gamma]\!]$ *for the* EPCF *typing context obtained by CPS-transforming the types of all the bindings of typing context* Γ.

Definition 21 (CPS Transformation of Effect Trees)**.** *Given an effect tree* $\mathbf{ET}(M)$ *in* $\mathsf{HEPCF}^{\mathsf{ATM}}$ *and a value* v*, the tree* $[\![\mathbf{ET}(M)]\!][\,v\,]$ *is defined coinductively as follows:*

$$\begin{aligned}
[\![\bot]\!][\,v\,] &\stackrel{\text{def}}{=} \bot \\
[\![\mathsf{return}\ V]\!][\,v\,] &\stackrel{\text{def}}{=} \mathbf{ET}(v\,[\![V]\!]) \\
[\![\sigma(c, \mathbf{ET}(M_1), \cdots, \mathbf{ET}(M_n))]\!][\,v\,] &\stackrel{\text{def}}{=} \sigma(c, [\![\mathbf{ET}(M_1)]\!][\,v\,], \cdots, [\![\mathbf{ET}(M_n)]\!][\,v\,])
\end{aligned}$$

$\boxed{[\![T]\!]}$ **for value types**

$$
\begin{aligned}
[\![B]\!] &\stackrel{\text{def}}{=} B \\
[\![E]\!] &\stackrel{\text{def}}{=} E \\
[\![T \to C]\!] &\stackrel{\text{def}}{=} [\![T]\!] \to [\![C]\!]
\end{aligned}
$$

$\boxed{[\![C]\!]}$ **for computation types**

$$
[\![\Sigma \triangleright T \,/\, A^{\text{ini}} \Rightarrow A^{\text{fin}}]\!] \stackrel{\text{def}}{=} [\![\Sigma]\!][\,([\![T]\!] \to [\![A^{\text{ini}}]\!]) \to [\![A^{\text{fin}}]\!]\,]
$$

$\boxed{[\![\Sigma]\!][\,\tau\,]}$ **for operation signatures**

$$
\begin{aligned}
[\![\emptyset]\!][\,\tau\,] &\stackrel{\text{def}}{=} \tau \\
[\![\Sigma \uplus \{\sigma : T^{\text{par}} \rightsquigarrow T^{\text{ari}} \,/\, A^{\text{ini}} \Rightarrow A^{\text{fin}}\}]\!][\,\tau\,] &\stackrel{\text{def}}{=} ([\![T^{\text{par}}]\!] \to ([\![T^{\text{ari}}]\!] \to [\![A^{\text{ini}}]\!]) \to [\![A^{\text{fin}}]\!]) \to [\![\Sigma]\!][\,\tau\,] \\
&\qquad \text{(where } \sigma \text{ is lower than any operation in } \Sigma\text{)}
\end{aligned}
$$

$\boxed{[\![V]\!]}$ **for values**

$$
\begin{aligned}
[\![x]\!] &\stackrel{\text{def}}{=} x \\
[\![c]\!] &\stackrel{\text{def}}{=} c \\
[\![\underline{\mathsf{n}}]\!] &\stackrel{\text{def}}{=} \underline{\mathsf{n}} \\
[\![\lambda x.M]\!] &\stackrel{\text{def}}{=} \lambda x.\mathsf{return}\,[\![M]\!] \\
[\![\mathsf{fix}\,x.\,V]\!] &\stackrel{\text{def}}{=} \mathsf{fix}\,x.[\![V]\!]
\end{aligned}
$$

$\boxed{[\![M]\!]}$ **for terms**

$$
[\![M]\!] \stackrel{\text{def}}{=} \lambda\overline{h}, k.[\![M]\!][\,\overline{h} \mid k\,]
$$

$\boxed{[\![M]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]}$ **for terms with handlers and continuations**

$$
\begin{aligned}
[\![\mathsf{return}\,V]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] &\stackrel{\text{def}}{=} v^{\mathsf{k}}\,[\![V]\!] \\
[\![\mathsf{let}\,x = M_1\,\mathsf{in}\,M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] &\stackrel{\text{def}}{=} [\![M_1]\!][\,\overline{v^{\mathsf{h}}} \mid \lambda x.[\![M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]\,] \\
[\![V_1\ V_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] &\stackrel{\text{def}}{=} [\![V_1]\!]\,[\![V_2]\!]\,\overline{v^{\mathsf{h}}}\,v^{\mathsf{k}} \\
[\![\mathsf{case}(V; M_1, \cdots, M_n)]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] &\stackrel{\text{def}}{=} \mathsf{case}([\![V]\!]; [\![M_1]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,], \cdots, [\![M_n]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]) \\
[\![\sigma_i(V; x.\,M)]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] &\stackrel{\text{def}}{=} v_i^{\mathsf{h}}\,[\![V]\!]\,\lambda x, \overline{h}, k.[\![M]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]\,\overline{h}\,k \\
&\qquad \text{(if } \sigma_i \text{ is the } i\text{-th operation in } \Sigma \text{ and } \sigma_i : T_i^{\text{par}} \rightsquigarrow T_i^{\text{ari}} \,/\, C_i^{\text{ini}} \Rightarrow A_i^{\text{fin}} \in \Sigma \text{ and } |C_i^{\text{ini}}.\Sigma| = |\overline{h}|\text{)} \\
[\![\sigma_i(V; x.\,M)]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] &\stackrel{\text{def}}{=} v_i^{\mathsf{h}}\,[\![V]\!]\,\lambda x.[\![M]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] \\
&\qquad \text{(if } \sigma_i \text{ is the } i\text{-th operation in } \Sigma \text{ and } \sigma_i : T_i^{\text{par}} \rightsquigarrow T_i^{\text{ari}} \,/\, T_i^{\text{ini}} \Rightarrow A_i^{\text{fin}} \in \Sigma\text{)} \\
[\![\mathsf{with}\,H\,\mathsf{handle}\,M]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] &\stackrel{\text{def}}{=} [\![M]\!][\,\lambda x_1, k_1.\mathsf{return}\,[\![M_1]\!], \cdots, \lambda x_n, k_n.\mathsf{return}\,[\![M_n]\!] \mid \lambda x.\mathsf{return}\,[\![M']\!]\,]\,\overline{v^{\mathsf{h}}}\,v^{\mathsf{k}} \\
&\qquad \text{(where } H = \{\mathsf{return}\,x \mapsto M'\} \uplus \{\sigma_i(x_i; k_i) \mapsto M_i\}^{1 \le i \le n}\text{)}
\end{aligned}
$$

Figure 5: CPS transformation. In the definition of $[\![M]\!]$ and $[\![M]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]$, we assume that M is well typed with an operation signature Σ. Furthermore, for sequences $\overline{h}$ in the definition of $[\![M]\!]$ and $\overline{v^{\mathsf{h}}}$ in $[\![M]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]$, $|\overline{h}| = |\overline{v^{\mathsf{h}}}| = |\Sigma|$.

3 Proofs

3.1 Type Soundness of $\mathsf{HEPCF}^{\mathsf{ATM}}$

Lemma 1 (Weakening). *Assume that $dom(\Gamma_2) \cap dom(\Gamma_1, \Gamma_3)$ is empty.*

- *If $\Gamma_1, \Gamma_3 \vdash V : T$, then $\Gamma_1, \Gamma_2, \Gamma_3 \vdash V : T$.*
- *If $\Gamma_1, \Gamma_3 \vdash M : C$, then $\Gamma_1, \Gamma_2, \Gamma_3 \vdash M : C$.*

Proof. Straightforward by mutual induction on the typing derivations. □

Lemma 2 (Value Substitution). *Assume that $\Gamma_1 \vdash V_0 : T_0$.*

- *If $\Gamma_1, x : T_0, \Gamma_2 \vdash V : T$, then $\Gamma_1, \Gamma_2 \vdash V[V_0/x] : T$.*
- *If $\Gamma_1, x : T_0, \Gamma_2 \vdash M : C$, then $\Gamma_1, \Gamma_2 \vdash M[V_0/x] : C$.*

Proof. Straightforward by mutual induction on the typing derivations. The case for (HT_Var) rests on Lemma 1. □

Lemma 3 (Canonical Forms). *Assume that $\emptyset \vdash V : T$.*

- *If $T = B$, then $V = c$ for some c such that $ty(c) = B$.*
- *If $T = \mathsf{n}$, then $V = \underline{\mathsf{i}}$ for some i such that $0 < i \leq n$.*
- *If $T = T' \to C'$, then $V = \lambda x.M$ for some x and M, or $V = \mathsf{fix}\, x.V'$ for some x and V'.*

Proof. Straightforward by case analysis on the typing derivation. Note that, for any c, $ty(c) = B$ for some B by Assumption 1. □

Lemma 4 (Progress). *If $\emptyset \vdash M : C$, then one of the following holds:*

- *$M = \mathsf{return}\ V$ for some V;*
- *$M = \sigma(V; x. M')$ for some σ, V, x, and M'; or*
- *$M \longrightarrow M'$ for some M'.*

Proof. By induction on the typing derivation applied last to derive $\emptyset \vdash M : C$.

Case (HT_Return): Obvious.

Case (HT_Let): We are given

$$\frac{\emptyset \vdash M_1 : \Sigma \triangleright T_1 \,/\, A \Rightarrow A_1 \quad x : T_1 \vdash M_2 : \Sigma \triangleright T_2 \,/\, A_2 \Rightarrow A}{\emptyset \vdash \mathsf{let}\, x = M_1 \,\mathsf{in}\, M_2 : \Sigma \triangleright T_2 \,/\, A_2 \Rightarrow A_1}$$

for some x, M_1, M_2, Σ, T_1, T_2, A_1, A_2, and A such that $M = (\mathsf{let}\, x = M_1 \,\mathsf{in}\, M_2)$ and $C = \Sigma \triangleright T_2 \,/\, A_2 \Rightarrow A_1$. By case analysis on the result of the IH on $\emptyset \vdash M_1 : \Sigma \triangleright T_1 \,/\, A \Rightarrow A_1$.

Case $\exists\, V_1.\ M_1 = \mathsf{return}\ V_1$: By (HE_LetV).
Case $\exists\, \sigma, V_1, y, M_1'.\ M_1 = \sigma(V_1; y. M_1')$: By (HE_LetOp).
Case $\exists\, M_1'.\ M_1 \longrightarrow M_1'$: By (HE_LetE).

Case (HT_App): We are given

$$\frac{\emptyset \vdash V_1 : T \to C \quad \emptyset \vdash V_2 : T}{\emptyset \vdash V_1\ V_2 : C}$$

for some V_1, V_2, and T such that $M = V_1\, V_2$. By case analysis on the result of applying Lemma 3 to $\emptyset \vdash V_1 : T \to C$.

Case $\exists\, x, M_1.\ V_1 = \lambda x.M_1$: By (HE_Beta).

Case $\exists\, x, V_1'.\ V_1 = \mathsf{fix}\, x.\, V_1'$: By (HE_Fix).

Case (HT_Case): We are given

$$\frac{\emptyset \vdash V : \mathsf{n} \quad \forall\, i \in [1, n].\ \emptyset \vdash M_i : C}{\emptyset \vdash \mathsf{case}(V; M_1, \cdots, M_n) : C}$$

for some V, n, $M_1, \cdots, M_n$ such that $M = \mathsf{case}(V; M_1, \cdots, M_n)$. By Lemma 3, $V = \underline{i}$ for some i such that $0 < i \leq n$. Thus, we have the conclusion by (HE_Case).

Case (HT_Op): Obvious.

Case (HT_Handle): We are given

$$\frac{\begin{array}{c} H = \{\mathsf{return}\, x \mapsto M_0\} \uplus \{\sigma_i(x_i; k_i) \mapsto M_i\}^{1 \leq i \leq n} \qquad \Sigma = \{\sigma_i : T_i^{\mathrm{par}} \rightsquigarrow T_i^{\mathrm{ari}} \,/\, C_i^{\mathrm{ini}} \Rrightarrow C_i^{\mathrm{fin}}\}^{1 \leq i \leq n} \\ \emptyset \vdash M' : \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rrightarrow C \qquad x : T \vdash M_0 : C^{\mathrm{ini}} \qquad \forall\, i \in [1, n].\ x_i : T_i^{\mathrm{par}}, k_i : T_i^{\mathrm{ari}} \to C_i^{\mathrm{ini}} \vdash M_i : C_i^{\mathrm{fin}} \end{array}}{\emptyset \vdash \mathsf{with}\, H\, \mathsf{handle}\, M' : C}$$

for some H, M', x, $M_0, \sigma_1, \cdots, \sigma_n$, $x_1 \cdots, x_n$ $k_1 \cdots, k_n$, $M_1, \cdots, M_n$, Σ, $\sigma_1, \cdots, \sigma_n$, $T_1^{\mathrm{par}}, \cdots, T_n^{\mathrm{par}}$, $T_1^{\mathrm{ari}}, \cdots, T_n^{\mathrm{ari}}$, $C_1^{\mathrm{ini}}, \cdots, C_n^{\mathrm{ini}}$, $C_1^{\mathrm{fin}}, \cdots, C_n^{\mathrm{fin}}$, T, and C^{ini} such that $M = \mathsf{with}\, H\, \mathsf{handle}\, M'$. By case analysis on the result of the IH on $\emptyset \vdash M' : \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rrightarrow C$.

Case $\exists\, V'.\ M' = \mathsf{return}\ V'$: By (HE_HandleV).

Case $\exists \sigma, V', y, M''.\ M' = \sigma(V'; y.\, M'')$: By the inversion of $\emptyset \vdash \sigma(V'; y.\, M'') : \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rrightarrow C$, we have $\sigma = \sigma_i$ for some i. Then, we have the conclusion by (HE_HandleOp).

Case $\exists\, M''.\ M' \longrightarrow M''$: By (HE_HandleE).

∎

Lemma 5 (Subject Reduction)**.** *If $\Gamma \vdash M : C$ and $M \longrightarrow M'$, then $\Gamma \vdash M' : C$.*

Proof. By induction on the typing derivation.

Case (HT_Return): We have $M = \mathsf{return}\ V$ for some V, but there is a contradiction because there is no evaluation rule applicable to $\mathsf{return}\ V$.

Case (HT_Let): We are given

$$\frac{\Gamma \vdash M_1 : \Sigma \triangleright T_1 \,/\, A \Rrightarrow A_1 \quad \Gamma, x : T_1 \vdash M_2 : \Sigma \triangleright T_2 \,/\, A_2 \Rrightarrow A}{\Gamma \vdash \mathsf{let}\, x = M_1\, \mathsf{in}\, M_2 : \Sigma \triangleright T_2 \,/\, A_2 \Rrightarrow A_1}$$

for some x, M_1, M_2, Σ, T_1, T_2, A_1, A_2, and A such that $M = (\mathsf{let}\, x = M_1\, \mathsf{in}\, M_2)$ and $C = \Sigma \triangleright T_2 \,/\, A_2 \Rrightarrow A_1$. We have $\mathsf{let}\, x = M_1\, \mathsf{in}\, M_2 \longrightarrow M'$. By case analysis on the evaluation rule applied last to derive it.

Case (HE_LetV): We are given

$$\mathsf{let}\, x = \mathsf{return}\ V_1\, \mathsf{in}\, M_2 \longrightarrow M_2[V_1/x]$$

for some V_1 such that $M_1 = \mathsf{return}\ V_1$ and $M' = M_2[V_1/x]$. Because $\Gamma \vdash \mathsf{return}\ V_1 : \Sigma \triangleright T_1 \,/\, A \Rrightarrow A_1$, its inversion implies $\Gamma \vdash V_1 : T_1$ and $A = A_1$. Thus, $\Gamma, x : T_1 \vdash M_2 : \Sigma \triangleright T_2 \,/\, A_2 \Rrightarrow A_1$. By Lemma 2, we have the conclusion $\Gamma \vdash M_2[V_1/x] : \Sigma \triangleright T_2 \,/\, A_2 \Rrightarrow A_1$.

Case (HE_LetOp): We are given

$$\mathsf{let}\, x = \sigma(V_1; y.\, M_1')\, \mathsf{in}\, M_2 \longrightarrow \sigma(V_1; y.\, \mathsf{let}\, x = M_1'\, \mathsf{in}\, M_2)$$

for some σ, V_1, y, and M_1' such that $M_1 = \sigma(V_1; y.\, M_1')$ and $M' = \sigma(V_1; y.\, \mathsf{let}\, x = M_1'\, \mathsf{in}\, M_2)$ and $y \notin \mathit{fv}(M_2)$. Because $\Gamma \vdash \sigma(V_1; y.\, M_1') : \Sigma \triangleright T_1 \,/\, A \Rrightarrow A_1$, its inversion implies

- $\sigma : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} \,/\, A^{\mathrm{ini}} \Rrightarrow A^{\mathrm{fin}} \in \Sigma$,

- $A_1 = A^{\mathrm{fin}}$,
- $\Gamma \vdash V_1 : T^{\mathrm{par}}$, and
- $\Gamma, y : T^{\mathrm{ari}} \vdash M'_1 : \Sigma \triangleright T_1 \,/\, A \Rightarrow A^{\mathrm{ini}}$

for some T^{par}, T^{ari}, A^{ini}, and A^{fin}. By Lemma 1, $\Gamma, y : T^{\mathrm{ari}}, x : T_1 \vdash M_2 : \Sigma \triangleright T_2 \,/\, A_2 \Rightarrow A$. By (HT_Let),

$$\Gamma, y : T^{\mathrm{ari}} \vdash \mathsf{let}\, x = M'_1 \,\mathsf{in}\, M_2 : \Sigma \triangleright T_2 \,/\, A_2 \Rightarrow A^{\mathrm{ini}} \ .$$

By (HT_Op) with $A_1 = A^{\mathrm{fin}}$, we have the conclusion

$$\Gamma \vdash \sigma(V_1; y.\, \mathsf{let}\, x = M'_1 \,\mathsf{in}\, M_2) : \Sigma \triangleright T_2 \,/\, A_2 \Rightarrow A_1 \ .$$

Case (HE_LetE): We are given

$$M_1 \longrightarrow M'_1$$

for some M'_1 such that $M' = (\mathsf{let}\, x = M'_1 \,\mathsf{in}\, M_2)$. By the IH, $\Gamma \vdash M'_1 : \Sigma \triangleright T_1 \,/\, A \Rightarrow A_1$. Therefore, by (HT_Let), we have the conclusion

$$\Gamma \vdash \mathsf{let}\, x = M'_1 \,\mathsf{in}\, M_2 : \Sigma \triangleright T_2 \,/\, A_2 \Rightarrow A_1.$$

Case (HT_App): We are given

$$\frac{\Gamma \vdash V_1 : T \to C \quad \Gamma \vdash V_2 : T}{\Gamma \vdash V_1\ V_2 : C}$$

for some V_1, V_2, and T such that $M = V_1\ V_2$. We have $V_1\ V_2 \longrightarrow M'$. By case analysis on the evaluation rule applied last to derive it.

Case (HE_Beta): We are given

$$(\lambda x.M_1)\ V_2 \longrightarrow M_1[V_2/x]$$

for some x and M_1 such that $V_1 = \lambda x.M_1$ and $M' = M_1[V_2/x]$. By the inversion of $\Gamma \vdash \lambda x.M_1 : T \to C$, we have $\Gamma, x : T \vdash M_1 : C$. Because $\Gamma \vdash V_2 : T$, we have the conclusion $\Gamma \vdash M_1[V_2/x] : C$ by Lemma 2.

Case (HE_Fix): We are given

$$(\mathsf{fix}\, x.\, V'_1)\ V_2 \longrightarrow V'_1[\mathsf{fix}\, x.\, V'_1/x]\ V_2$$

for some x and V'_1 such that $V_1 = \mathsf{fix}\, x.\, V'_1$ and $M' = V'_1[\mathsf{fix}\, x.\, V'_1/x]\ V_2$. By the inversion of $\Gamma \vdash \mathsf{fix}\, x.\, V'_1 : T \to C$, we have $\Gamma, x : T \to C \vdash V'_1 : T \to C$. By Lemma 2, $\Gamma \vdash V'_1[\mathsf{fix}\, x.\, V'_1/x] : T \to C$. Therefore, by (HT_App), we have the conclusion

$$\Gamma \vdash V'_1[\mathsf{fix}\, x.\, V'_1/x]\ V_2 : C \ .$$

Case (HT_Case): We are given

$$\frac{\Gamma \vdash V : \mathsf{n} \quad \forall i \in [1, n].\ \Gamma \vdash M_i : C}{\Gamma \vdash \mathsf{case}(V; M_1, \cdots, M_n) : C}$$

for some V, n, $M_1, \cdots, M_n$ such that $M = \mathsf{case}(V; M_1, \cdots, M_n)$. Because $\mathsf{case}(V; M_1, \cdots, M_n) \longrightarrow M'$, we have $V = \underline{i}$ and $M' = M_i$ for some i such that $0 < i \le n$. Because $\Gamma \vdash M_i : C$, we have the conclusion.

Case (HT_Op): We have $M = \sigma(V; x.\, M'')$ for some σ, V, x, and M'', but there is a contradiction because there is no evaluation rule applicable to $\sigma(V; x.\, M'')$.

Case (HT_Handle): We are given

$$\frac{\begin{array}{c} H = \{\mathsf{return}\, x \mapsto M_0\} \uplus \{\sigma_i(x_i; k_i) \mapsto M_i\}^{1 \le i \le n} \qquad \Sigma = \{\sigma_i : T_i^{\mathrm{par}} \rightsquigarrow T_i^{\mathrm{ari}} \,/\, C_i^{\mathrm{ini}} \Rightarrow C_i^{\mathrm{fin}}\}^{1 \le i \le n} \\ \Gamma \vdash M'_0 : \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C \qquad \Gamma, x : T \vdash M_0 : C^{\mathrm{ini}} \qquad \forall i \in [1, n].\ \Gamma, x_i : T_i^{\mathrm{par}}, k_i : T_i^{\mathrm{ari}} \to C_i^{\mathrm{ini}} \vdash M_i : C_i^{\mathrm{fin}} \end{array}}{\Gamma \vdash \mathsf{with}\, H \,\mathsf{handle}\, M'_0 : C}$$

for some H, M'_0, x, M_0, $\sigma_1, \cdots, \sigma_n$, $x_1 \cdots, x_n$ $k_1 \cdots, k_n$, $M_1, \cdots, M_n$, Σ, $\sigma_1, \cdots, \sigma_n$, $T_1^{\mathrm{par}}, \cdots, T_n^{\mathrm{par}}$, $T_1^{\mathrm{ari}}, \cdots, T_n^{\mathrm{ari}}$, $C_1^{\mathrm{ini}}, \cdots, C_n^{\mathrm{ini}}$, $C_1^{\mathrm{fin}}, \cdots, C_n^{\mathrm{fin}}$, T, and C^{ini} such that $M = \mathsf{with}\, H \,\mathsf{handle}\, M'_0$. We have $\mathsf{with}\, H \,\mathsf{handle}\, M'_0 \longrightarrow M'$. By case analysis on the evaluation rule applied last to derive it.

Case (HE_HandleV): We are given

$$\mathsf{with}\ H\ \mathsf{handle}\ \mathsf{return}\ V \longrightarrow M_0[V/x]$$

for some V such that $M_0' = \mathsf{return}\ V$ and $M' = M_0[V/x]$. By the inversion of $\Gamma \vdash \mathsf{return}\ V : \Sigma \triangleright T \mathbin{/} C^{\mathrm{ini}} \Rightarrow C$, we have $\Gamma \vdash V : T$ and $C^{\mathrm{ini}} = C$. By Lemma 2 with $\Gamma, x : T \vdash M_0 : C^{\mathrm{ini}}$, we have the conclusion $\Gamma \vdash M_0[V/x] : C$.

Case (HE_HandleOp): We are given

$$\mathsf{with}\ H\ \mathsf{handle}\ \sigma_i(V; y.\, M_0'') \longrightarrow M_i[V/x_i][\lambda y.\mathsf{with}\ H\ \mathsf{handle}\ M_0''/k_i]$$

for some i, V, y, and M_0'' such that $M_0' = \sigma_i(V; y.\, M_0'')$ and $M' = M_i[V/x_i][\lambda y.\mathsf{with}\ H\ \mathsf{handle}\ M_0''/k_i]$. By the inversion of $\Gamma \vdash \sigma_i(V; y.\, M_0'') : \Sigma \triangleright T \mathbin{/} C^{\mathrm{ini}} \Rightarrow C$, we have

- $C = C_i^{\mathrm{fin}}$,
- $\Gamma \vdash V : T_i^{\mathrm{par}}$,and
- $\Gamma, y : T_i^{\mathrm{ari}} \vdash M_0'' : \Sigma \triangleright T \mathbin{/} C^{\mathrm{ini}} \Rightarrow C_i^{\mathrm{ini}}$.

By Lemma 1,

- $\Gamma, y : T_i^{\mathrm{ari}}, x : T \vdash M_0 : C^{\mathrm{ini}}$ and
- $\forall j \in [1, n].\ \Gamma, y : T_i^{\mathrm{ari}}, x_i : T_i^{\mathrm{par}}, k_i : T_i^{\mathrm{ari}} \to C_i^{\mathrm{ini}} \vdash M_i : C_i^{\mathrm{fin}}$.

Therefore, by (HT_Handle) and (HT_Abs),

$$\Gamma \vdash \lambda y.\mathsf{with}\ H\ \mathsf{handle}\ M_0'' : T_i^{\mathrm{ari}} \to C_i^{\mathrm{ini}} \ .$$

Thus, by Lemma 2 and $C = C_i^{\mathrm{fin}}$, we have the conclusion

$$\Gamma \vdash M_i[V/x_i][\lambda y.\mathsf{with}\ H\ \mathsf{handle}\ M_0''/k_i] : C \ .$$

Case (HE_HandleE): We are given $M_0' \longrightarrow M_0''$ for some M_0'' such that $M' = \mathsf{with}\ H\ \mathsf{handle}\ M_0''$. By the IH, $\Gamma \vdash M_0'' : \Sigma \triangleright T \mathbin{/} C^{\mathrm{ini}} \Rightarrow C$. By (HT_Handle), we have the conclusion

$$\Gamma \vdash \mathsf{with}\ H\ \mathsf{handle}\ M_0'' : C \ .$$

□

3.2 Type Soundness of EPCF

Lemma 6 (Weakening). *Assume that $dom(\Delta_2) \cap dom(\Delta_1, \Delta_3)$ is empty.*

- *If $\Xi \parallel \Delta_1, \Delta_3 \vdash v : \tau$, then $\Xi \parallel \Delta_1, \Delta_2, \Delta_3 \vdash v : \tau$.*
- *If $\Xi \parallel \Delta_1, \Delta_3 \vdash e : \tau$, then $\Xi \parallel \Delta_1, \Delta_2, \Delta_3 \vdash e : \tau$.*

Proof. Straightforward by mutual induction on the typing derivations. □

Lemma 7 (Value Substitution). *Assume that $\Xi \parallel \Delta_1 \vdash v_0 : \tau_0$.*

- *If $\Xi \parallel \Delta_1, x : \tau_0, \Delta_2 \vdash v : \tau$, then $\Xi \parallel \Delta_1, \Delta_2 \vdash v[v_0/x] : \tau$.*
- *If $\Xi \parallel \Delta_1, x : \tau_0, \Delta_2 \vdash e : \tau$, then $\Xi \parallel \Delta_1, \Delta_2 \vdash e[v_0/x] : \tau$.*

Proof. Straightforward by mutual induction on the typing derivations. The case for (T_Var) rests on Lemma 6. □

Lemma 8 (Canonical Forms). *Assume that $\Xi \parallel \emptyset \vdash v : \tau$.*

- *If $\tau = B$, then $v = c$ for some c such that $ty(c) = B$.*
- *If $\tau = \mathsf{n}$, then $v = \underline{\mathsf{i}}$ for some i such that $0 < i \leq n$.*

- *If* $\tau = \tau_1 \to \tau_2$*, then* $v = \lambda x.e$ *for some* x *and* e*, or* $v = \mathsf{fix}\, x.v'$ *for some* x *and* v'*.*

Proof. Straightforward by case analysis on the typing derivation. Note that, for any c, $ty(c) = B$ for some B by Assumption 1. □

Lemma 9 (Progress)**.** *If* $\Xi \parallel \emptyset \vdash e : \tau$*, then one of the following holds:*

- $e = \mathsf{return}\, v$ *for some* v*;*
- $e = \sigma(v; x.\, e')$ *for some* σ*,* v*,* x*, and* e'*; or*
- $e \longrightarrow e'$ *for some* e'*.*

Proof. By induction on the typing derivation applied last to derive $\Xi \parallel \emptyset \vdash e : \tau$.

Case (T_Return): Obvious.

Case (T_Let): We are given

$$\frac{\Xi \parallel \emptyset \vdash e_1 : \tau_1 \quad \Xi \parallel x : \tau_1 \vdash e_2 : \tau}{\Xi \parallel \emptyset \vdash \mathsf{let}\, x = e_1 \,\mathsf{in}\, e_2 : \tau}$$

for some x, e_1, e_2, and τ_1 such that $e = (\mathsf{let}\, x = e_1 \,\mathsf{in}\, e_2)$. By case analysis on the result of the IH on $\Xi \parallel \emptyset \vdash e_1 : \tau_1$.

Case $\exists\, v_1.\ e_1 = \mathsf{return}\, v_1$: By (E_LetV).

Case $\exists\, \sigma, v_1, y, e_1'.\ e_1 = \sigma(v_1; y.\, e_1')$: By (E_LetOp).

Case $\exists\, e_1'.\ e_1 \longrightarrow e_1'$: By (E_LetE).

Case (T_App): We are given

$$\frac{\Xi \parallel \emptyset \vdash v_1 : \tau' \to \tau \quad \Xi \parallel \emptyset \vdash v_2 : \tau'}{\Xi \parallel \emptyset \vdash v_1\, v_2 : \tau}$$

for some v_1, v_2, and τ' such that $e = v_1\, v_2$. By case analysis on the result of applying Lemma 8 to $\Xi \parallel \emptyset \vdash v_1 : \tau' \to \tau$.

Case $\exists\, x, e_1.\ v_1 = \lambda x.e_1$: By (E_Beta).

Case $\exists\, x, v_1'.\ v_1 = \mathsf{fix}\, x.v_1'$: By (E_Fix).

Case (T_Case): We are given

$$\frac{\Xi \parallel \emptyset \vdash v : \mathsf{n} \quad \forall\, i \in [1, n].\ \Xi \parallel \emptyset \vdash e_i : \tau}{\Xi \parallel \emptyset \vdash \mathsf{case}(v; e_1, \cdots, e_n) : \tau}$$

for some v, n, $e_1, \cdots, e_n$ such that $e = \mathsf{case}(v; e_1, \cdots, e_n)$. By Lemma 8, $v = \underline{\mathsf{i}}$ for some i such that $0 < i \leq n$. Thus, we have the conclusion by (E_Case).

Case (T_Op): Obvious.

□

Lemma 10 (Subject Reduction)**.** *If* $\Xi \parallel \Delta \vdash e : \tau$ *and* $e \longrightarrow e'$*, then* $\Xi \parallel \Delta \vdash e' : \tau$*.*

Proof. By induction on the typing derivation.

Case (T_Return): We have $e = \mathsf{return}\, v$ for some v, but there is a contradiction because there is no evaluation rule applicable to $\mathsf{return}\, v$.

Case (T_Let): We are given

$$\frac{\Xi \parallel \Delta \vdash e_1 : \tau_1 \quad \Xi \parallel \Delta, x : \tau_1 \vdash e_2 : \tau}{\Xi \parallel \Delta \vdash \mathsf{let}\, x = e_1 \,\mathsf{in}\, e_2 : \tau}$$

for some x, e_1, e_2, and τ_1 such that $e = (\mathsf{let}\, x = e_1 \,\mathsf{in}\, e_2)$. We have $\mathsf{let}\, x = e_1 \,\mathsf{in}\, e_2 \longrightarrow e'$. By case analysis on the evaluation rule applied last to derive it.

Case (E_LetV): We are given

$$\mathsf{let}\, x = \mathsf{return}\, v_1 \,\mathsf{in}\, e_2 \longrightarrow e_2[v_1/x]$$

for some v_1 such that $e_1 = \mathsf{return}\, v_1$ and $e' = e_2[v_1/x]$. Because $\Xi \parallel \Delta \vdash \mathsf{return}\, v_1 : \tau_1$, its inversion implies $\Xi \parallel \Delta \vdash v_1 : \tau_1$. By Lemma 7, we have the conclusion $\Xi \parallel \Delta \vdash e_2[v_1/x] : \tau$.

Case (E_LetOp): We are given

$$\mathsf{let}\, x = \sigma(v_1; y.\, e'_1) \,\mathsf{in}\, e_2 \longrightarrow \sigma(v_1; y.\, \mathsf{let}\, x = e'_1 \,\mathsf{in}\, e_2)$$

for some σ, v_1, y, and e'_1 such that $e_1 = \sigma(v_1; y.\, e'_1)$ and $e' = \sigma(v_1; y.\, \mathsf{let}\, x = e'_1 \,\mathsf{in}\, e_2)$ and $y \notin fv(e_2)$. Because $\Xi \parallel \Delta \vdash \sigma(v_1; y.\, e'_1) : \tau_1$, its inversion implies

- $\sigma : B \rightsquigarrow E \in \Xi$,
- $\Xi \parallel \Delta \vdash v_1 : B$, and
- $\Xi \parallel \Delta, y : E \vdash e'_1 : \tau_1$

for some B and E. By Lemma 6, $\Xi \parallel \Delta, y : E, x : \tau_1 \vdash e_2 : \tau$. By (T_Let),

$$\Xi \parallel \Delta, y : E \vdash \mathsf{let}\, x = e'_1 \,\mathsf{in}\, e_2 : \tau \ .$$

By (T_Op), we have the conclusion

$$\Xi \parallel \Delta \vdash \sigma(v_1; y.\, \mathsf{let}\, x = e'_1 \,\mathsf{in}\, e_2) : \tau \ .$$

Case (E_LetE): We are given

$$e_1 \longrightarrow e'_1$$

for some e'_1 such that $e' = (\mathsf{let}\, x = e'_1 \,\mathsf{in}\, e_2)$. By the IH, $\Xi \parallel \Delta \vdash e'_1 : \tau_1$. Therefore, by (T_Let), we have the conclusion

$$\Xi \parallel \Delta \vdash \mathsf{let}\, x = e'_1 \,\mathsf{in}\, e_2 : \tau.$$

Case (T_App): We are given

$$\frac{\Xi \parallel \Delta \vdash v_1 : \tau' \to \tau \quad \Xi \parallel \Delta \vdash v_2 : \tau'}{\Xi \parallel \Delta \vdash v_1\, v_2 : \tau}$$

for some v_1, v_2, and τ' such that $e = v_1\, v_2$. We have $v_1\, v_2 \longrightarrow e'$. By case analysis on the evaluation rule applied last to derive it.

Case (E_Beta): We are given

$$(\lambda x.e_1)\, v_2 \longrightarrow e_1[v_2/x]$$

for some x and e_1 such that $v_1 = \lambda x.e_1$ and $e' = e_1[v_2/x]$. By the inversion of $\Xi \parallel \Delta \vdash \lambda x.e_1 : \tau' \to \tau$, we have $\Xi \parallel \Delta, x : \tau' \vdash e_1 : \tau$. Because $\Xi \parallel \Delta \vdash v_2 : \tau'$, we have the conclusion $\Xi \parallel \Delta \vdash e_1[v_2/x] : \tau$ by Lemma 7.

Case (E_Fix): We are given

$$(\mathsf{fix}\, x.v'_1)\, v_2 \longrightarrow v'_1[\mathsf{fix}\, x.v'_1/x]\, v_2$$

for some x and v'_1 such that $v_1 = \mathsf{fix}\, x.v'_1$ and $e' = v'_1[\mathsf{fix}\, x.v'_1/x]\, v_2$. By the inversion of $\Xi \parallel \Delta \vdash \mathsf{fix}\, x.v'_1 : \tau' \to \tau$, we have $\Xi \parallel \Delta, x : \tau' \to \tau \vdash v'_1 : \tau' \to \tau$. By Lemma 7, $\Xi \parallel \Delta \vdash v'_1[\mathsf{fix}\, x.v'_1/x] : \tau' \to \tau$. Therefore, by (T_App), we have the conclusion

$$\Xi \parallel \Delta \vdash v'_1[\mathsf{fix}\, x.v'_1/x]\, v_2 : \tau \ .$$

Case (T_CASE): We are given

$$\frac{\Xi \parallel \Delta \vdash v : \mathsf{n} \quad \forall i \in [1,n].\ \Xi \parallel \Delta \vdash e_i : \tau}{\Xi \parallel \Delta \vdash \mathsf{case}(v; e_1, \cdots, e_n) : \tau}$$

for some v, n, $e_1, \cdots, e_n$ such that $e = \mathsf{case}(v; e_1, \cdots, e_n)$. Because $\mathsf{case}(v; e_1, \cdots, e_n) \longrightarrow e'$, we have $v = \underline{i}$ and $e' = e_i$ for some i such that $0 < i \leq n$. Because $\Xi \parallel \Delta \vdash e_i : \tau$, we have the conclusion.

Case (T_OP): We have $e = \sigma(v; x.\, e'')$ for some σ, v, x, and e'', but there is a contradiction because there is no evaluation rule applicable to $\sigma(v; x.\, e'')$.

□

3.3 Type Preservation

Definition 22 (Pre-Order on Typing Contexts)**.** *We write $\Delta_1 \preceq \Delta_2$ if $dom(\Delta_1) \subseteq dom(\Delta_2)$ and, for any $x \in dom(\Delta_1)$, $\Delta_1(x) = \Delta_2(x)$.*

Definition 23 (Typing of Effect Handlers)**.** *Let $\Sigma = \{\sigma_i : T_i^{\mathrm{par}} \rightsquigarrow T_i^{\mathrm{ari}} \,/\, A_i^{\mathrm{ini}} \Rightarrow A_i^{\mathrm{fin}}\}^{1 \leq i \leq n}$ for some $\sigma_1, \cdots, \sigma_n$, $T_1^{\mathrm{par}}, \cdots, T_n^{\mathrm{par}}$, $T_1^{\mathrm{ari}}, \cdots, T_n^{\mathrm{ari}}$, $A_1^{\mathrm{ini}}, \cdots, A_n^{\mathrm{ini}}$, and $A_1^{\mathrm{fin}}, \cdots, A_n^{\mathrm{fin}}$ such that $\sigma_1, \cdots, \sigma_n$ are ordered. For a variable sequence $\overline{h} = h_1, \cdots, h_n$, we write $\overline{h} : \Sigma$ to denote the typing context that, for each $i \in [1,n]$, assigns to the variable h_i the type $[\![T_i^{\mathrm{par}}]\!] \rightarrow ([\![T_i^{\mathrm{ari}}]\!] \rightarrow [\![A_i^{\mathrm{ini}}]\!]) \rightarrow [\![A_i^{\mathrm{fin}}]\!]$. For a value sequence $\overline{v^{\mathsf{h}}} = v_1^{\mathsf{h}}, \cdots, v_n^{\mathsf{h}}$, we write $\Xi \parallel \Delta \vdash \overline{v^{\mathsf{h}}} : \Sigma$ if, for each $i \in [1,n]$, $\Xi \parallel \Delta \vdash v_i^{\mathsf{h}} : [\![T_i^{\mathrm{par}}]\!] \rightarrow ([\![T_i^{\mathrm{ari}}]\!] \rightarrow [\![A_i^{\mathrm{ini}}]\!]) \rightarrow [\![A_i^{\mathrm{fin}}]\!]$ holds.*

Lemma 11 (Type Preservation of the CPS Transformation)**.** *Assume that $[\![\Gamma]\!] \preceq \Delta$.*

- *If $\Gamma \vdash V : T$, then $\Xi \parallel \Delta \vdash [\![V]\!] : [\![T]\!]$ for any Ξ.*
- *If $\Gamma \vdash M : \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}$ and $\Xi \parallel \Delta \vdash \overline{v^{\mathsf{h}}} : \Sigma$ and $\Xi \parallel \Delta \vdash v^{\mathsf{k}} : [\![T]\!] \rightarrow [\![A^{\mathrm{ini}}]\!]$, then $\Xi \parallel \Delta \vdash [\![M]\!][\, \overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] : [\![A^{\mathrm{fin}}]\!]$.*
- *If $\Gamma \vdash M : C$, then $\Xi \parallel \Delta \vdash [\![M]\!] : [\![C]\!]$ for any Ξ.*

Proof. By mutual induction on the typing derivations.

- Assume that $\Gamma \vdash V : T$ is given. By case analysis on the typing rule applied last to derive it.

  Case (HT_VAR): Obvious by (T_VAR).

  Case (HT_CONST): Obvious by (T_CONST). Note that $ty(c)$ is base type by Assumption 1.

  Case (HT_ECONST): Obvious by (T_ECONST).

  Case (HT_ABS): We are given $\Gamma \vdash \lambda x.M : T' \rightarrow C'$ for some x, M, T', and C' such that $V = \lambda x.M$ and $T = T' \rightarrow C'$. By inversion, $\Gamma, x : T' \vdash M : C'$. Because $[\![\Gamma]\!] \preceq \Delta$, we have $[\![\Gamma]\!], x : [\![T']\!] \preceq \Delta, x : [\![T']\!]$. Therefore, by the IH, $\Xi \parallel \Delta, x : [\![T']\!] \vdash [\![M]\!] : [\![C']\!]$. By (T_RETURN) and (T_ABS), $\Xi \parallel \Delta \vdash \lambda x.\mathsf{return}\, [\![M]\!] : [\![T']\!] \rightarrow [\![C']\!]$. By the definition of the CPS transformation, we have the conclusion.

  Case (HT_FIX): We are given $\Gamma \vdash \mathsf{fix}\, x.\, V' : T' \rightarrow C'$ for some x, V', T', and C' such that $V = \mathsf{fix}\, x.\, V'$ and $T = T' \rightarrow C'$. By inversion, $\Gamma, x : T' \rightarrow C' \vdash V' : T' \rightarrow C'$. Because $[\![\Gamma]\!] \preceq \Delta$, we have $[\![\Gamma]\!], x : [\![T']\!] \rightarrow [\![C']\!] \preceq \Delta, x : [\![T']\!] \rightarrow [\![C']\!]$. Therefore, by the IH, $\Xi \parallel \Delta, x : [\![T']\!] \rightarrow [\![C']\!] \vdash [\![V']\!] : [\![T']\!] \rightarrow [\![C']\!]$. By (T_FIX), $\Xi \parallel \Delta \vdash \mathsf{fix}\, x.[\![V']\!] : [\![T']\!] \rightarrow [\![C']\!]$. By the definition of the CPS transformation, we have the conclusion.

- Assume that $\Gamma \vdash M : \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}$ is given. Let $\Sigma = \{\sigma_i : T_i^{\mathrm{par}} \rightsquigarrow T_i^{\mathrm{ari}} \,/\, A_i^{\mathrm{ini}} \Rightarrow A_i^{\mathrm{fin}}\}^{1 \leq i \leq n}$ for some $\sigma_1, \cdots, \sigma_n$, $T_1^{\mathrm{par}}, \cdots, T_n^{\mathrm{par}}$, $T_1^{\mathrm{ari}}, \cdots, T_n^{\mathrm{ari}}$, $A_1^{\mathrm{ini}}, \cdots, A_n^{\mathrm{ini}}$, and $A_1^{\mathrm{fin}}, \cdots, A_n^{\mathrm{fin}}$ such that $\sigma_1, \cdots, \sigma_n$ are ordered. By case analysis on the typing rule applied last to derive it.

  Case (HT_RETURN): We are given

$$\frac{\Gamma \vdash V : T}{\Gamma \vdash \mathsf{return}\ V : \Sigma \triangleright T \,/\, A \Rightarrow A}$$

for some V and A such that $M = \mathsf{return}\ V$ and $A^{\mathrm{ini}} = A^{\mathrm{fin}} = A$. By the IH, $\Xi \parallel \Delta \vdash [\![V]\!] : [\![T]\!]$. By the definition, it suffices to show that

$$\Xi \parallel \Delta \vdash v^{\mathsf{k}}\,[\![V]\!] : [\![A]\!]\ ,$$

which is derived as follows:

$$\frac{\Xi \parallel \Delta \vdash v^{\mathsf{k}} : [\![T]\!] \to [\![A]\!] \qquad \Xi \parallel \Delta \vdash [\![V]\!] : [\![T]\!]}{\Xi \parallel \Delta \vdash v^{\mathsf{k}}\,[\![V]\!] : [\![A]\!]}\ (\text{T\_App})$$

Case (HT_Let): We are given

$$\frac{\Gamma \vdash M_1 : \Sigma \rhd T_1 \,/\, A \Rightarrow A^{\mathrm{fin}} \quad \Gamma, x : T_1 \vdash M_2 : \Sigma \rhd T \,/\, A^{\mathrm{ini}} \Rightarrow A}{\Gamma \vdash \mathsf{let}\, x = M_1\, \mathsf{in}\, M_2 : \Sigma \rhd T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}}$$

for some x, M_1, M_2, T_1, and A such that $M = (\mathsf{let}\, x = M_1\, \mathsf{in}\, M_2)$. Without loss of generality, we can assume that $x \notin dom(\Delta)$. By the definition, it suffices to show that

$$\Xi \parallel \Delta \vdash [\![M_1]\!][\,\overline{v^{\mathsf{h}}} \mid \lambda x.[\![M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]\,] : [\![A^{\mathrm{fin}}]\!]\ .$$

Because $[\![\Gamma]\!] \preceq \Delta$, we have $[\![\Gamma]\!], x : [\![T_1]\!] \preceq \Delta, x : [\![T_1]\!]$. By Lemma 6, $\Xi \parallel \Delta, x : [\![T_1]\!] \vdash \overline{v^{\mathsf{h}}} : \Sigma$ and $\Xi \parallel \Delta, x : [\![T_1]\!] \vdash v^{\mathsf{k}} : [\![T]\!] \to [\![A^{\mathrm{ini}}]\!]$. Therefore, by the IH on $\Gamma, x : T_1 \vdash M_2 : \Sigma \rhd T \,/\, A^{\mathrm{ini}} \Rightarrow A$, we have

$$\Xi \parallel \Delta, x : [\![T_1]\!] \vdash [\![M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] : [\![A]\!]\ .$$

By (T_Abs),

$$\Xi \parallel \Delta \vdash \lambda x.[\![M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] : [\![T_1]\!] \to [\![A]\!]\ .$$

Therefore, by the IH on $\Gamma \vdash M_1 : \Sigma \rhd T_1 \,/\, A \Rightarrow A^{\mathrm{fin}}$, we have the conclusion

$$\Xi \parallel \Delta \vdash [\![M_1]\!][\,\overline{v^{\mathsf{h}}} \mid \lambda x.[\![M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]\,] : [\![A^{\mathrm{fin}}]\!]\ .$$

Case (HT_App): We are given

$$\frac{\Gamma \vdash V_1 : T' \to \Sigma \rhd T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}} \quad \Gamma \vdash V_2 : T'}{\Gamma \vdash V_1\ V_2 : \Sigma \rhd T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}}$$

for some V_1, V_2, and T' such that $M = V_1\ V_2$. By the definition, it suffices to show that

$$\Xi \parallel \Delta \vdash [\![V_1]\!]\,[\![V_2]\!]\,\overline{v^{\mathsf{h}}}\,v^{\mathsf{k}} : [\![A^{\mathrm{fin}}]\!]\ .$$

Then, it suffices to show that

$$\Xi \parallel \Delta \vdash [\![V_1]\!]\,[\![V_2]\!] : [\![\Sigma]\!][\,([\![T]\!] \to [\![A^{\mathrm{ini}}]\!]) \to [\![A^{\mathrm{fin}}]\!]\,]\ ,$$

which is derived by the IHs and (T_App).

Case (HT_Case): We are given

$$\frac{\Gamma \vdash V : \mathsf{n} \quad \forall\, i \in [1, n].\ \Gamma \vdash M_i : \Sigma \rhd T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}}{\Gamma \vdash \mathsf{case}(V; M_1, \cdots, M_n) : \Sigma \rhd T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}}$$

for some V_1, $M_1, \cdots, M_n$, and n such that $M = \mathsf{case}(V; M_1, \cdots, M_n)$. By the definition, it suffices to show that

$$\Xi \parallel \Delta \vdash \mathsf{case}([\![V]\!]; [\![M_1]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,], \cdots, [\![M_n]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]) : [\![A^{\mathrm{fin}}]\!]\ .$$

By the IHs,

- $\Xi \parallel \Delta \vdash [\![V]\!] : \mathsf{n}$ and
- $\forall\, i \in [1, n].\ \Xi \parallel \Delta \vdash [\![M_i]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] : [\![A^{\mathrm{fin}}]\!]$.

Therefore, by (T_Case), we have the conclusion.

Case (HT_Op): We are given

$$\frac{\Gamma \vdash V' : T_i^{\mathrm{par}} \quad \Gamma, x : T_i^{\mathrm{ari}} \vdash M' : \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A_i^{\mathrm{ini}}}{\Gamma \vdash \sigma_i(V'; x.\, M') : \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A_i^{\mathrm{fin}}}$$

for some V', x, M', and i such that $M = \sigma_i(V'; x.\, M')$ and $A^{\mathrm{fin}} = A_i^{\mathrm{fin}}$. By case analysis on A_i^{ini}.

Case $\exists\, C_i^{\mathrm{ini}}.\ A_i^{\mathrm{ini}} = C_i^{\mathrm{ini}}$: By the definition, it suffices to show that

$$\Xi \parallel \Delta \vdash v_i^{\mathsf{h}}\, [\![V']\!]\, \lambda x, \overline{h}, k. [\![M']\!][\, \overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]\, \overline{h}\, k : [\![A_i^{\mathrm{fin}}]\!]$$

where $|\overline{h}| = |C_i^{\mathrm{ini}}.\Sigma|$. Let $C_i^{\mathrm{ini}} = \Sigma' \triangleright T' \,/\, C'^{\mathrm{ini}} \Rightarrow C'^{\mathrm{fin}}$ for some Σ', T', C'^{ini}, and C'^{fin}. Let $\Delta' = \Delta, x : [\![T_i^{\mathrm{ari}}]\!], \overline{h} : \Sigma', k : [\![T']\!] \to [\![C'^{\mathrm{ini}}]\!]$. Because $[\![\Gamma]\!] \preceq \Delta$, we have $[\![\Gamma]\!], x : [\![T_i^{\mathrm{ari}}]\!] \preceq \Delta'$. By Lemma 6, $\Xi \parallel \Delta' \vdash \overline{v^{\mathsf{h}}} : \Sigma$ and $\Xi \parallel \Delta' \vdash v^{\mathsf{k}} : [\![T]\!] \to [\![A^{\mathrm{ini}}]\!]$. Therefore, by the IHs,

- $\Xi \parallel \Delta \vdash [\![V']\!] : [\![T_i^{\mathrm{par}}]\!]$ and
- $\Xi \parallel \Delta' \vdash [\![M']\!][\, \overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] : [\![C_i^{\mathrm{ini}}]\!]$.

By (T_Abs), (T_App), (T_Var), (T_Let), and (T_Return),

$$\Xi \parallel \Delta \vdash \lambda x, \overline{h}, k. [\![M']\!][\, \overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]\, \overline{h}\, k : [\![T_i^{\mathrm{ari}}]\!] \to [\![C_i^{\mathrm{ini}}]\!] \ .$$

The conclusion is derived as follows:

$$\mathcal{D} = \frac{\Xi \parallel \Delta \vdash v_i^{\mathsf{h}} : [\![T_i^{\mathrm{par}}]\!] \to ([\![T_i^{\mathrm{ari}}]\!] \to [\![A_i^{\mathrm{ini}}]\!]) \to [\![A_i^{\mathrm{fin}}]\!] \qquad \Xi \parallel \Delta \vdash [\![V']\!] : [\![T_i^{\mathrm{par}}]\!]}{\Xi \parallel \Delta \vdash v_i^{\mathsf{h}}\, [\![V']\!] : ([\![T_i^{\mathrm{ari}}]\!] \to [\![A_i^{\mathrm{ini}}]\!]) \to [\![A_i^{\mathrm{fin}}]\!]}\ (\text{T\_App})$$

$$\frac{\mathcal{D} \qquad \Xi \parallel \Delta \vdash \lambda x, \overline{h}, k. [\![M']\!][\, \overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]\, \overline{h}\, k : [\![T_i^{\mathrm{ari}}]\!] \to [\![C_i^{\mathrm{ini}}]\!] \qquad C_i^{\mathrm{ini}} = A_i^{\mathrm{ini}}}{\Xi \parallel \Delta \vdash v_i^{\mathsf{h}}\, [\![V']\!]\, \lambda x, \overline{h}, k. [\![M']\!][\, \overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]\, \overline{h}\, k : [\![A_i^{\mathrm{fin}}]\!]}\ (\text{T\_Let}),(\text{T\_App})$$

Case $\exists\, T_i^{\mathrm{ini}}.\ A_i^{\mathrm{ini}} = T_i^{\mathrm{ini}}$: By the definition, it suffices to show that

$$\Xi \parallel \Delta \vdash v_i^{\mathsf{h}}\, [\![V']\!]\, \lambda x. [\![M']\!][\, \overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] : [\![A_i^{\mathrm{fin}}]\!] \ .$$

Because $[\![\Gamma]\!] \preceq \Delta$, we have $[\![\Gamma]\!], x : [\![T_i^{\mathrm{ari}}]\!] \preceq \Delta, x : [\![T_i^{\mathrm{ari}}]\!]$. By Lemma 6, $\Xi \parallel \Delta, x : [\![T_i^{\mathrm{ari}}]\!] \vdash \overline{v^{\mathsf{h}}} : \Sigma$ and $\Xi \parallel \Delta, x : [\![T_i^{\mathrm{ari}}]\!] \vdash v^{\mathsf{k}} : [\![T]\!] \to [\![A^{\mathrm{ini}}]\!]$. Therefore, by the IHs,

- $\Xi \parallel \Delta \vdash [\![V']\!] : [\![T_i^{\mathrm{par}}]\!]$ and
- $\Xi \parallel \Delta, x : [\![T_i^{\mathrm{ari}}]\!] \vdash [\![M']\!][\, \overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] : [\![A_i^{\mathrm{ini}}]\!]$.

By (T_Abs),

$$\Xi \parallel \Delta \vdash \lambda x. [\![M']\!][\, \overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] : [\![T_i^{\mathrm{ari}}]\!] \to [\![A_i^{\mathrm{ini}}]\!] \ .$$

The conclusion is derived as follows:

$$\mathcal{D} = \frac{\Xi \parallel \Delta \vdash v_i^{\mathsf{h}} : [\![T_i^{\mathrm{par}}]\!] \to ([\![T_i^{\mathrm{ari}}]\!] \to [\![A_i^{\mathrm{ini}}]\!]) \to [\![A_i^{\mathrm{fin}}]\!] \qquad \Xi \parallel \Delta \vdash [\![V']\!] : [\![T_i^{\mathrm{par}}]\!]}{\Xi \parallel \Delta \vdash v_i^{\mathsf{h}}\, [\![V']\!] : ([\![T_i^{\mathrm{ari}}]\!] \to [\![A_i^{\mathrm{ini}}]\!]) \to [\![A_i^{\mathrm{fin}}]\!]}\ (\text{T\_App})$$

$$\frac{\mathcal{D} \qquad \Xi \parallel \Delta \vdash \lambda x. [\![M']\!][\, \overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] : [\![T_i^{\mathrm{ari}}]\!] \to [\![A_i^{\mathrm{ini}}]\!]}{\Xi \parallel \Delta \vdash v_i^{\mathsf{h}}\, [\![V']\!]\, \lambda x. [\![M']\!][\, \overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] : [\![A_i^{\mathrm{fin}}]\!]}\ (\text{T\_Let}),(\text{T\_App})$$

Case (HT_Handle): We are given

$$\frac{\begin{array}{c} H' = \{\mathsf{return}\, x \mapsto M_0\} \uplus \{\sigma'_j(x'_j; k'_j) \mapsto M'_j\}^{1 \le j \le m} \\ \Sigma' = \{\sigma'_j : T'^{\mathrm{par}}_j \rightsquigarrow T'^{\mathrm{ari}}_j \,/\, C'^{\mathrm{ini}}_j \Rightarrow C'^{\mathrm{fin}}_j\}^{1 \le j \le m} \\ \Gamma \vdash M' : \Sigma' \triangleright T' \,/\, C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}} \quad \Gamma, x : T' \vdash M_0 : C^{\mathrm{ini}} \\ \forall\, j \in [1, m].\ \Gamma, x'_j : T'^{\mathrm{par}}_j, k'_j : T'^{\mathrm{ari}}_j \to C'^{\mathrm{ini}}_j \vdash M'_j : C'^{\mathrm{fin}}_j \end{array}}{\Gamma \vdash \mathsf{with}\, H'\, \mathsf{handle}\, M' : \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}}$$

for some H', M', x, M_0, $\sigma'_1, \cdots, \sigma'_m$, $x'_1, \cdots, x'_m$, $k'_1, \cdots, k'_m$, $M'_1, \cdots, M'_m$, ${T'}^{\mathrm{par}}_1, \cdots, {T'}^{\mathrm{par}}_m$, ${T'}^{\mathrm{ari}}_1, \cdots, {T'}^{\mathrm{ari}}_m$, ${C'}^{\mathrm{ini}}_1, \cdots, {C'}^{\mathrm{ini}}_m$, and ${C'}^{\mathrm{fin}}_1, \cdots, {C'}^{\mathrm{fin}}_m$, Σ', T', C^{ini}, and C^{fin} such that $M = \mathsf{with}\ H'\ \mathsf{handle}\ M'$ and $C^{\mathrm{fin}} = \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}$. By the definition, it suffices to show that

$$\Xi \parallel \Delta \vdash [\![M']\!][\, \lambda x'_1, k'_1.\mathsf{return}\, [\![M'_1]\!], \cdots, \lambda x'_m, k'_m.\mathsf{return}\, [\![M'_m]\!] \mid \lambda x.\mathsf{return}\, [\![M_0]\!] \,]\, \overline{v^{\mathsf{h}}}\, v^{\mathsf{k}} : [\![A^{\mathrm{fin}}]\!] \ .$$

Because $[\![\Gamma]\!] \preceq \Delta$, we have $[\![\Gamma]\!], x : [\![T']\!] \preceq \Delta, x : [\![T']\!]$ and, for each $j \in [1, m]$, $[\![\Gamma]\!], x'_j : [\![{T'}^{\mathrm{par}}_j]\!], k'_j : [\![{T'}^{\mathrm{ari}}_j]\!] \to [\![{C'}^{\mathrm{ini}}_j]\!] \preceq \Delta, x'_j : [\![{T'}^{\mathrm{par}}_j]\!], k'_j : [\![{T'}^{\mathrm{ari}}_j]\!] \to [\![{C'}^{\mathrm{ini}}_j]\!]$. Therefore, by the IHs on the typing derivations of $M_0, M'_1, \cdots, M'_m$,

- $\Xi \parallel \Delta, x : [\![T']\!] \vdash [\![M_0]\!] : [\![C^{\mathrm{ini}}]\!]$ and
- $\forall\, j \in [1, m].\ \Xi \parallel \Delta, x'_j : [\![{T'}^{\mathrm{par}}_j]\!], k'_j : [\![{T'}^{\mathrm{ari}}_j]\!] \to [\![{C'}^{\mathrm{ini}}_j]\!] \vdash [\![M'_j]\!] : [\![{C'}^{\mathrm{fin}}_j]\!]$.

By (T_Abs) and (T_Return),

- $\Xi \parallel \Delta \vdash \lambda x.\mathsf{return}\, [\![M_0]\!] : [\![T']\!] \to [\![C^{\mathrm{ini}}]\!]$ and
- $\forall\, j \in [1, m].\ \Xi \parallel \Delta \vdash \lambda x'_j, k'_j.\mathsf{return}\, [\![M'_j]\!] : [\![{T'}^{\mathrm{par}}_j]\!] \to ([\![{T'}^{\mathrm{ari}}_j]\!] \to [\![{C'}^{\mathrm{ini}}_j]\!]) \to [\![{C'}^{\mathrm{fin}}_j]\!]$.

Therefore,

$$\Xi \parallel \Delta \vdash \lambda x'_1, k'_1.\mathsf{return}\, [\![M'_1]\!], \cdots, \lambda x'_m, k'_m.\mathsf{return}\, [\![M'_m]\!] : \Sigma' \ .$$

By the IH on the typing derivation of M',

$$\Xi \parallel \Delta \vdash [\![M']\!][\, \lambda x'_1, k'_1.\mathsf{return}\, [\![M'_1]\!], \cdots, \lambda x'_m, k'_m.\mathsf{return}\, [\![M'_m]\!] \mid \lambda x.\mathsf{return}\, [\![M_0]\!] \,] : [\![C^{\mathrm{fin}}]\!] \ .$$

Because $C^{\mathrm{fin}} = \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}$ and $\Xi \parallel \Delta \vdash \overline{v^{\mathsf{h}}} : \Sigma$ and $\Xi \parallel \Delta \vdash v^{\mathsf{k}} : [\![T]\!] \to [\![A^{\mathrm{ini}}]\!]$, we have the conclusion.

- Assume that $\Gamma \vdash M : \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}$ is given. Let $\overline{h}$ be a variable sequence such that $|\overline{h}| = |\Sigma|$. Then, by the definition of the CPS transformation, it suffices to show that $\Xi \parallel \Delta, \overline{h} : \Sigma, k : [\![T]\!] \to [\![A^{\mathrm{ini}}]\!] \vdash [\![M]\!][\, \overline{h} \mid k \,] : [\![A^{\mathrm{fin}}]\!]$, which is shown by case (11).

□

3.4 Semantics Preservation

Lemma 12 (Substitution is a Homomorphism). *For any V' and x, the following holds:*

1. *For any M, $\overline{v^{\mathsf{h}}}$, and v^{k}, $[\![M]\!][\, \overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}} \,][[\![V']\!]/x] = [\![M[\,V'/x]]\!][\, \overline{v^{\mathsf{h}}[[\![V']\!]/x]} \mid v^{\mathsf{k}}[[\![V']\!]/x] \,]$.*
2. *For any M, $[\![M]\!][[\![V']\!]/x] = [\![M[\,V'/x]]\!]$.*
3. *For any V, $[\![V]\!][[\![V']\!]/x] = [\![V[\,V'/x]]\!]$.*

Proof. By mutual induction on M and V.

1. By case analysis on M.

   Case $\exists\, V.\ M = \mathsf{return}\ V$: The conclusion is shown as follows:

$$\begin{array}{ll} & [\![M]\!][\, \overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}} \,][[\![V']\!]/x] \\ = & [\![\mathsf{return}\ V]\!][\, \overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}} \,][[\![V']\!]/x] \\ = & v^{\mathsf{k}}[[\![V']\!]/x]\ [\![V]\!][[\![V']\!]/x] \\ = & v^{\mathsf{k}}[[\![V']\!]/x]\ [\![V[\,V'/x]]\!] \quad \text{(by the IH)} \\ = & [\![\mathsf{return}\ V[\,V'/x]]\!][\, \overline{v^{\mathsf{h}}[[\![V']\!]/x]} \mid v^{\mathsf{k}}[[\![V']\!]/x] \,] \\ = & [\![M[\,V'/x]]\!][\, \overline{v^{\mathsf{h}}[[\![V']\!]/x]} \mid v^{\mathsf{k}}[[\![V']\!]/x] \,] \ . \end{array}$$

Case $\exists\, M_1, M_2, y.\ M = (\mathsf{let}\, y = M_1 \,\mathsf{in}\, M_2)$: Without loss of generality, we can assume that $y \notin fv(V') \cup \{x\}$. Then, the conclusion is shown as follows:

$$\begin{aligned}
& \llbracket M \rrbracket [\, \overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}} \,][\llbracket V' \rrbracket / x] \\
= \;& \llbracket \mathsf{let}\, y = M_1 \,\mathsf{in}\, M_2 \rrbracket [\, \overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}} \,][\llbracket V' \rrbracket / x] \\
= \;& \llbracket M_1 \rrbracket [\, \overline{v^{\mathsf{h}}} \mid \lambda y. \llbracket M_2 \rrbracket [\, \overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}} \,] \,][\llbracket V' \rrbracket / x] \\
= \;& \llbracket M_1 [\, V'/x] \rrbracket [\, \overline{v^{\mathsf{h}}[\llbracket V' \rrbracket / x]} \mid (\lambda y. \llbracket M_2 \rrbracket [\, \overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}} \,])[\llbracket V' \rrbracket / x] \,] && \text{(by the IH on } M_1\text{)} \\
= \;& \llbracket M_1 [\, V'/x] \rrbracket [\, \overline{v^{\mathsf{h}}[\llbracket V' \rrbracket / x]} \mid \lambda y. \llbracket M_2 \rrbracket [\, \overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}} \,][\llbracket V' \rrbracket / x] \,] \\
= \;& \llbracket M_1 [\, V'/x] \rrbracket [\, \overline{v^{\mathsf{h}}[\llbracket V' \rrbracket / x]} \mid \lambda y. \llbracket M_2 [\, V'/x] \rrbracket [\, \overline{v^{\mathsf{h}}[\llbracket V' \rrbracket / x]} \mid v^{\mathsf{k}}[\llbracket V' \rrbracket / x] \,] \,] && \text{(by the IH on } M_2\text{)} \\
= \;& \llbracket \mathsf{let}\, y = M_1 [\, V'/x] \,\mathsf{in}\, M_2 [\, V'/x] \rrbracket [\, \overline{v^{\mathsf{h}}[\llbracket V' \rrbracket / x]} \mid v^{\mathsf{k}}[\llbracket V' \rrbracket / x] \,] \\
= \;& \llbracket M [\, V'/x] \rrbracket [\, \overline{v^{\mathsf{h}}[\llbracket V' \rrbracket / x]} \mid v^{\mathsf{k}}[\llbracket V' \rrbracket / x] \,]\ .
\end{aligned}$$

Case $\exists\, V_1, V_2.\ M = V_1\ V_2$: The conclusion is shown as follows:

$$\begin{aligned}
& \llbracket M \rrbracket [\, \overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}} \,][\llbracket V' \rrbracket / x] \\
= \;& \llbracket V_1\ V_2 \rrbracket [\, \overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}} \,][\llbracket V' \rrbracket / x] \\
= \;& (\llbracket V_1 \rrbracket\, \llbracket V_2 \rrbracket\, \overline{v^{\mathsf{h}}}\, v^{\mathsf{k}})[\llbracket V' \rrbracket / x] \\
= \;& \llbracket V_1 [\, V'/x] \rrbracket\, \llbracket V_2 [\, V'/x] \rrbracket\, \overline{v^{\mathsf{h}}[\llbracket V' \rrbracket / x]}\, v^{\mathsf{k}}[\llbracket V' \rrbracket / x] && \text{(by the IHs on } V_1 \text{ and } V_2\text{)} \\
= \;& \llbracket V_1 [\, V'/x]\ V_2 [\, V'/x] \rrbracket [\, \overline{v^{\mathsf{h}}[\llbracket V' \rrbracket / x]} \mid v^{\mathsf{k}}[\llbracket V' \rrbracket / x] \,] \\
= \;& \llbracket M [\, V'/x] \rrbracket [\, \overline{v^{\mathsf{h}}[\llbracket V' \rrbracket / x]} \mid v^{\mathsf{k}}[\llbracket V' \rrbracket / x] \,]\ .
\end{aligned}$$

Case $\exists\, V, M_1, \cdots, M_n.\ M = \mathsf{case}(\, V; M_1, \cdots, M_n)$: The conclusion is shown as follows:

$$\begin{aligned}
& \llbracket M \rrbracket [\, \overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}} \,][\llbracket V' \rrbracket / x] \\
= \;& \llbracket \mathsf{case}(\, V; M_1, \cdots, M_n) \rrbracket [\, \overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}} \,][\llbracket V' \rrbracket / x] \\
= \;& (\mathsf{case}(\llbracket V \rrbracket; \llbracket M_1 \rrbracket [\, \overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}} \,], \cdots, \llbracket M_n \rrbracket [\, \overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}} \,]))[\llbracket V' \rrbracket / x] \\
= \;& \mathsf{case}(\llbracket V \rrbracket [\llbracket V' \rrbracket / x]; \llbracket M_1 \rrbracket [\, \overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}} \,][\llbracket V' \rrbracket / x], \cdots, \llbracket M_n \rrbracket [\, \overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}} \,][\llbracket V' \rrbracket / x]) \\
= \;& \mathsf{case}(\llbracket V [\, V'/x] \rrbracket; \llbracket M_1 [\, V'/x] \rrbracket [\, \overline{v^{\mathsf{h}}[\llbracket V' \rrbracket / x]} \mid v^{\mathsf{k}}[\llbracket V' \rrbracket / x] \,], \cdots, \llbracket M_n [\, V'/x] \rrbracket [\, \overline{v^{\mathsf{h}}[\llbracket V' \rrbracket / x]} \mid v^{\mathsf{k}}[\llbracket V' \rrbracket / x] \,]) \\
& \quad \text{(by the IHs on } V, M_1, \cdots, M_n\text{)} \\
= \;& \llbracket \mathsf{case}(\, V [\, V'/x]; M_1 [\, V'/x], \cdots, M_n [\, V'/x]) \rrbracket [\, \overline{v^{\mathsf{h}}[\llbracket V' \rrbracket / x]} \mid v^{\mathsf{k}}[\llbracket V' \rrbracket / x] \,] \\
= \;& \llbracket \mathsf{case}(\, V; M_1, \cdots, M_n)[\, V'/x] \rrbracket [\, \overline{v^{\mathsf{h}}[\llbracket V' \rrbracket / x]} \mid v^{\mathsf{k}}[\llbracket V' \rrbracket / x] \,] \\
= \;& \llbracket M [\, V'/x] \rrbracket [\, \overline{v^{\mathsf{h}}[\llbracket V' \rrbracket / x]} \mid v^{\mathsf{k}}[\llbracket V' \rrbracket / x] \,]\ .
\end{aligned}$$

Case $\exists \sigma_i, V, y, M'.\ M = \sigma_i(\, V; y.\, M')$: Without loss of generality, we can assume that $y \notin fv(V') \cup \{x\}$. Furthermore, assume that $\overline{v^{\mathsf{h}}}$ includes a value v^{h}_i corresponding to σ_i. Then, the conclusion is shown as follows:

$$\begin{aligned}
& \llbracket M \rrbracket [\, \overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}} \,][\llbracket V' \rrbracket / x] \\
= \;& \llbracket \sigma_i(\, V; y.\, M') \rrbracket [\, \overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}} \,][\llbracket V' \rrbracket / x] \\
= \;& (v^{\mathsf{h}}_i\, \llbracket V \rrbracket\, \lambda y, \overline{h}, k. \llbracket M' \rrbracket [\, \overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}} \,]\, \overline{h}\, k)[\llbracket V' \rrbracket / x] \\
= \;& v^{\mathsf{h}}_i[\llbracket V' \rrbracket / x]\, \llbracket V \rrbracket [\llbracket V' \rrbracket / x]\, \lambda y, \overline{h}, k. \llbracket M' \rrbracket [\, \overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}} \,][\llbracket V' \rrbracket / x]\, \overline{h}\, k \\
= \;& v^{\mathsf{h}}_i[\llbracket V' \rrbracket / x]\, \llbracket V [\, V'/x] \rrbracket\, \lambda y, \overline{h}, k. \llbracket M' [\, V'/x] \rrbracket [\, \overline{v^{\mathsf{h}}[\llbracket V' \rrbracket / x]} \mid v^{\mathsf{k}}[\llbracket V' \rrbracket / x] \,]\, \overline{h}\, k && \text{(by the IHs on } V \text{ and } M'\text{)} \\
= \;& \llbracket \sigma_i(\, V [\, V'/x]; y.\, M' [\, V'/x]) \rrbracket [\, \overline{v^{\mathsf{h}}[\llbracket V' \rrbracket / x]} \mid v^{\mathsf{k}}[\llbracket V' \rrbracket / x] \,] \\
= \;& \llbracket \sigma_i(\, V; y.\, M')[\, V'/x] \rrbracket [\, \overline{v^{\mathsf{h}}[\llbracket V' \rrbracket / x]} \mid v^{\mathsf{k}}[\llbracket V' \rrbracket / x] \,] \\
= \;& \llbracket M [\, V'/x] \rrbracket [\, \overline{v^{\mathsf{h}}[\llbracket V' \rrbracket / x]} \mid v^{\mathsf{k}}[\llbracket V' \rrbracket / x] \,]
\end{aligned}$$

or

$$\begin{aligned}
& \llbracket M \rrbracket [\, \overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}} \,][\llbracket V' \rrbracket / x] \\
= \;& \llbracket \sigma_i(\, V; y.\, M') \rrbracket [\, \overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}} \,][\llbracket V' \rrbracket / x] \\
= \;& (v^{\mathsf{h}}_i\, \llbracket V \rrbracket\, \lambda y. \llbracket M' \rrbracket [\, \overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}} \,])[\llbracket V' \rrbracket / x] \\
= \;& v^{\mathsf{h}}_i[\llbracket V' \rrbracket / x]\, \llbracket V \rrbracket [\llbracket V' \rrbracket / x]\, \lambda y. \llbracket M' \rrbracket [\, \overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}} \,][\llbracket V' \rrbracket / x] \\
= \;& v^{\mathsf{h}}_i[\llbracket V' \rrbracket / x]\, \llbracket V [\, V'/x] \rrbracket\, \lambda y. \llbracket M' [\, V'/x] \rrbracket [\, \overline{v^{\mathsf{h}}[\llbracket V' \rrbracket / x]} \mid v^{\mathsf{k}}[\llbracket V' \rrbracket / x] \,] && \text{(by the IHs on } V \text{ and } M'\text{)} \\
= \;& \llbracket \sigma_i(\, V [\, V'/x]; y.\, M' [\, V'/x]) \rrbracket [\, \overline{v^{\mathsf{h}}[\llbracket V' \rrbracket / x]} \mid v^{\mathsf{k}}[\llbracket V' \rrbracket / x] \,] \\
= \;& \llbracket \sigma_i(\, V; y.\, M')[\, V'/x] \rrbracket [\, \overline{v^{\mathsf{h}}[\llbracket V' \rrbracket / x]} \mid v^{\mathsf{k}}[\llbracket V' \rrbracket / x] \,] \\
= \;& \llbracket M [\, V'/x] \rrbracket [\, \overline{v^{\mathsf{h}}[\llbracket V' \rrbracket / x]} \mid v^{\mathsf{k}}[\llbracket V' \rrbracket / x] \,]\ .
\end{aligned}$$

Note that the value substitution does not influence the operation signature used in typing M.

Case $\exists\, H, M'.\ M = \mathsf{with}\ H\ \mathsf{handle}\ M'$: Let $H = \{\mathsf{return}\, y \mapsto M''\} \uplus \{\sigma(y_i; k_i) \mapsto M_i\}^{1 \le i \le n}$ for some y, M'', $y_1, \cdots, y_n, k_1, \cdots, k_n, M_1, \cdots, M_n$. Without loss of generality, we can assume that $y, y_1, \cdots, y_n, k_1, \ldots, k_n$ are distinct from the variables in $\mathit{fv}(V') \cup \{x\}$. Then, the conclusion is shown as follows:

$$\begin{array}{rl} & [\![M]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,][[\![V']\!]/x] \\ = & [\![\mathsf{with}\ H\ \mathsf{handle}\ M']\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,][[\![V']\!]/x] \\ = & ([\![M']\!][\,\lambda y_1, k_1.\mathsf{return}\, [\![M_1]\!], \cdots, \lambda y_n, k_n.\mathsf{return}\, [\![M_n]\!] \mid \lambda y.\mathsf{return}\, [\![M'']\!]\,]\,\overline{v^{\mathsf{h}}}\, v^{\mathsf{k}})[[\![V']\!]/x] \\ = & [\![M'[V'/x]]\!][\,V_1, \cdots, V_n \mid \lambda y.\mathsf{return}\, [\![M''[V'/x]]\!]\,]\,\overline{v^{\mathsf{h}}[[\![V']\!]/x]}\, v^{\mathsf{k}}[[\![V']\!]/x] \\ & \quad (\text{where } V_i \stackrel{\text{def}}{=} \lambda y_i, k_i.\mathsf{return}\, [\![M_i[V'/x]]\!], \text{ by the IHs on } M', M'', M_1, \cdots, M_n) \\ = & [\![(\mathsf{with}\ H\ \mathsf{handle}\ M')[V'/x]]\!][\,\overline{v^{\mathsf{h}}[[\![V']\!]/x]} \mid v^{\mathsf{k}}[[\![V']\!]/x]\,] \\ = & [\![M[V'/x]]\!][\,\overline{v^{\mathsf{h}}[[\![V']\!]/x]} \mid v^{\mathsf{k}}[[\![V']\!]/x]\,]\ . \end{array}$$

2. By case (1).

3. By induction on V.

   Case $\exists\, y.\ V = y$: Obvious.

   Case $\exists\, c.\ V = c$: Obvious.

   Case $\exists\, \underline{\mathsf{n}}.\ V = \underline{\mathsf{n}}$: Obvious.

   Case $\exists\, y, M.\ V = \lambda y.M$: By the IH.

   Case $\exists\, y, V_0.\ V = \mathsf{fix}\, y.\, V_0$: By the IH.

□

Lemma 13 (Handler and Continuation Substitution). *If $x \notin \mathit{fv}(M)$, then*

$$[\![M]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,][v/x] = [\![M]\!][\,\overline{v^{\mathsf{h}}[v/x]} \mid v^{\mathsf{k}}[v/x]\,]\ .$$

Proof. By induction on M.

Case $\exists\, V'.\ M = \mathsf{return}\ V'$: The conclusion is shown as follows:

$$\begin{array}{rll} & [\![M]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,][v/x] & \\ = & [\![\mathsf{return}\ V']\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,][v/x] & \\ = & v^{\mathsf{k}}[v/x]\,[\![V']\!][v/x] & \\ = & v^{\mathsf{k}}[v/x]\,[\![V']\!] & (\text{Note that } x \text{ does not occur free in } [\![V']\!]) \\ = & [\![\mathsf{return}\ V']\!][\,\overline{v^{\mathsf{h}}[v/x]} \mid v^{\mathsf{k}}[v/x]\,] & \\ = & [\![M]\!][\,\overline{v^{\mathsf{h}}[v/x]} \mid v^{\mathsf{k}}[v/x]\,]\ . & \end{array}$$

Case $\exists\, M_1, M_2, y.\ M = (\mathsf{let}\, y = M_1\ \mathsf{in}\ M_2)$: Without loss of generality, we can assume that $y \notin \mathit{fv}(v) \cup \{x\}$. Then, the conclusion is shown as follows:

$$\begin{array}{rll} & [\![M]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,][v/x] & \\ = & [\![\mathsf{let}\, y = M_1\ \mathsf{in}\ M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,][v/x] & \\ = & [\![M_1]\!][\,\overline{v^{\mathsf{h}}} \mid \lambda y.[\![M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]\,][v/x] & \\ = & [\![M_1]\!][\,\overline{v^{\mathsf{h}}[v/x]} \mid (\lambda y.[\![M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,])[v/x]\,] & (\text{by the IH on } M_1) \\ = & [\![M_1]\!][\,\overline{v^{\mathsf{h}}[v/x]} \mid \lambda y.[\![M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,][v/x]\,] & \\ = & [\![M_1]\!][\,\overline{v^{\mathsf{h}}[v/x]} \mid \lambda y.[\![M_2]\!][\,\overline{v^{\mathsf{h}}[v/x]} \mid v^{\mathsf{k}}[v/x]\,]\,] & (\text{by the IH on } M_2) \\ = & [\![\mathsf{let}\, y = M_1\ \mathsf{in}\ M_2]\!][\,\overline{v^{\mathsf{h}}[v/x]} \mid v^{\mathsf{k}}[v/x]\,] & \\ = & [\![M]\!][\,\overline{v^{\mathsf{h}}[v/x]} \mid v^{\mathsf{k}}[v/x]\,]\ . & \end{array}$$

Case $\exists\, V_1, V_2.\ M = V_1\ V_2$: The conclusion is shown as follows:

$$\begin{array}{cl} & [\![M]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,][v/x] \\ = & [\![V_1\ V_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,][v/x] \\ = & ([\![V_1]\!]\,[\![V_2]\!]\,\overline{v^{\mathsf{h}}}\,v^{\mathsf{k}})[v/x] \\ = & [\![V_1]\!]\,[\![V_2]\!]\,\overline{v^{\mathsf{h}}[v/x]}\,v^{\mathsf{k}}[v/x] \quad \text{(Note that } x \text{ does not occur free in } [\![V_1]\!] \text{ nor } [\![V_2]\!]) \\ = & [\![V_1\ V_2]\!][\,\overline{v^{\mathsf{h}}[v/x]} \mid v^{\mathsf{k}}[v/x]\,] \\ = & [\![M]\!][\,\overline{v^{\mathsf{h}}[v/x]} \mid v^{\mathsf{k}}[v/x]\,]\ . \end{array}$$

Case $\exists\, V', M_1, \cdots, M_n.\ M = \mathsf{case}(V'; M_1, \cdots, M_n)$: The conclusion is shown as follows:

$$\begin{array}{cl} & [\![M]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,][v/x] \\ = & [\![\mathsf{case}(V'; M_1, \cdots, M_n)]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,][v/x] \\ = & (\mathsf{case}([\![V']\!]; [\![M_1]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,], \cdots, [\![M_n]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]))[v/x] \\ = & \mathsf{case}([\![V']\!][v/x]; [\![M_1]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,][v/x], \cdots, [\![M_n]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,][v/x]) \\ = & \mathsf{case}([\![V']\!]; [\![M_1]\!][\,\overline{v^{\mathsf{h}}[v/x]} \mid v^{\mathsf{k}}[v/x]\,], \cdots, [\![M_n]\!][\,\overline{v^{\mathsf{h}}[v/x]} \mid v^{\mathsf{k}}[v/x]\,]) \\ & \quad \text{(by the IHs on } M_1, \cdots, M_n\text{; note that } x \text{ does not occur free in } [\![V']\!]) \\ = & [\![\mathsf{case}(V'; M_1, \cdots, M_n)]\!][\,\overline{v^{\mathsf{h}}[v/x]} \mid v^{\mathsf{k}}[v/x]\,] \\ = & [\![M]\!][\,\overline{v^{\mathsf{h}}[v/x]} \mid v^{\mathsf{k}}[v/x]\,]\ . \end{array}$$

Case $\exists\, \sigma_i, V', y, M'.\ M = \sigma_i(V'; y.\ M')$: Without loss of generality, we can assume that $y \notin \mathit{fv}(v) \cup \{x\}$. Furthermore, assume that $\overline{v^{\mathsf{h}}}$ includes a value v^{h}_i corresponding to σ_i. Then, the conclusion is shown as follows:

$$\begin{array}{cl} & [\![M]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,][v/x] \\ = & [\![\sigma_i(V'; y.\ M')]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,][v/x] \\ = & (v^{\mathsf{h}}_i\,[\![V']\!]\,\lambda y, \overline{h}, k.[\![M']\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]\,\overline{h}\,k)[v/x] \\ = & v^{\mathsf{h}}_i[v/x]\,[\![V']\!][v/x]\,\lambda y, \overline{h}, k.[\![M']\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,][v/x]\,\overline{h}\,k \\ = & v^{\mathsf{h}}_i[v/x]\,[\![V']\!]\,\lambda y, \overline{h}, k.[\![M']\!][\,\overline{v^{\mathsf{h}}[v/x]} \mid v^{\mathsf{k}}[v/x]\,]\,\overline{h}\,k \quad \text{(by the IH on } M'\text{; note that } x \text{ does not occur free in } [\![V']\!]) \\ = & [\![\sigma_i(V'; y.\ M')]\!][\,\overline{v^{\mathsf{h}}[v/x]} \mid v^{\mathsf{k}}[v/x]\,] \\ = & [\![M]\!][\,\overline{v^{\mathsf{h}}[v/x]} \mid v^{\mathsf{k}}[v/x]\,] \end{array}$$

or

$$\begin{array}{cl} & [\![M]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,][v/x] \\ = & [\![\sigma_i(V'; y.\ M')]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,][v/x] \\ = & (v^{\mathsf{h}}_i\,[\![V']\!]\,\lambda y.[\![M']\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,])[v/x] \\ = & v^{\mathsf{h}}_i[v/x]\,[\![V']\!][v/x]\,\lambda y.[\![M']\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,][v/x] \\ = & v^{\mathsf{h}}_i[v/x]\,[\![V']\!]\,\lambda y.[\![M']\!][\,\overline{v^{\mathsf{h}}[v/x]} \mid v^{\mathsf{k}}[v/x]\,] \quad \text{(by the IH on } M'\text{; note that } x \text{ does not occur free in } [\![V']\!]) \\ = & [\![\sigma_i(V'; y.\ M')]\!][\,\overline{v^{\mathsf{h}}[v/x]} \mid v^{\mathsf{k}}[v/x]\,] \\ = & [\![M]\!][\,\overline{v^{\mathsf{h}}[v/x]} \mid v^{\mathsf{k}}[v/x]\,]\ . \end{array}$$

Case $\exists\, H, M'.\ M = \mathsf{with}\ H\ \mathsf{handle}\ M'$: Let $H = \{\mathsf{return}\, y \ \mapsto\ M''\} \uplus \{\sigma(y_i; k_i) \ \mapsto\ M_i\}^{1 \le i \le n}$ for some y, M'', $y_1, \cdots, y_n$, $k_1, \cdots, k_n$, $M_1, \cdots, M_n$. Without loss of generality, we can assume that $y, y_1, \cdots, y_n, k_1, \ldots, k_n$ are distinct from the variables in $\mathit{fv}(v) \cup \{x\}$. Then, the conclusion is shown as follows:

$$\begin{array}{cl} & [\![M]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,][v/x] \\ = & [\![\mathsf{with}\ H\ \mathsf{handle}\ M']\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,][v/x] \\ = & ([\![M']\!][\,\lambda y_1, k_1.\mathsf{return}\,[\![M_1]\!], \cdots, \lambda y_n, k_n.\mathsf{return}\,[\![M_n]\!] \mid \lambda y.\mathsf{return}\,[\![M'']\!]\,]\,\overline{v^{\mathsf{h}}}\,v^{\mathsf{k}})[v/x] \\ = & [\![M']\!][\,\lambda y_1, k_1.\mathsf{return}\,[\![M_1]\!], \cdots, \lambda y_n, k_n.\mathsf{return}\,[\![M_n]\!] \mid \lambda y.\mathsf{return}\,[\![M'']\!]\,]\,\overline{v^{\mathsf{h}}[v/x]}\,v^{\mathsf{k}}[v/x] \\ = & \quad \text{(by the IH on } M'\text{; note that } x \text{ does not occur free in } M_1, \cdots, M_n, M'') \\ = & [\![(\mathsf{with}\ H\ \mathsf{handle}\ M')]\!][\,\overline{v^{\mathsf{h}}[v/x]} \mid v^{\mathsf{k}}[v/x]\,] \\ = & [\![M]\!][\,\overline{v^{\mathsf{h}}[v/x]} \mid v^{\mathsf{k}}[v/x]\,]\ . \end{array}$$

$\square$

Lemma 14 (Simulation up to Reduction)**.** *Assume that* $\Gamma \vdash M : \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}$*. If* $M \longrightarrow M'$*, then, for any* $\overline{v^{\mathsf{h}}}$ *and* v^{k} *such that* $|\overline{v^{\mathsf{h}}}| = |\Sigma|$*, either of the following holds:*

- $M' \longrightarrow^{*} \sigma(V_0; x.\, M_0)$ *and* $[\![M]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] = [\![\sigma(V_0; x.\, M_0)]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]$ *for some* σ*,* V_0*,* x*, and* M_0*; or*
- $M' \longrightarrow^{*} M''$ *and* $[\![M]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] \longrightarrow^{+} [\![M'']\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]$ *for some* M''*.*

Proof. By case analysis on the evaluation rule applied to derive $M \longrightarrow M'$.

Case (HE_Beta): We are given

$$(\lambda x.M_1)\ V_2 \longrightarrow M_1[\,V_2/x]$$

for some x, M_1, and V_2 such that $M = (\lambda x.M_1)\ V_2$ and $M' = M_1[\,V_2/x]$. Because $\Gamma \vdash M : \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}$, we have the following derivation for some T':

$$\dfrac{\dfrac{\Gamma, x : T' \vdash M_1 : \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}}{\Gamma \vdash \lambda x.M_1 : T' \to \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}}\ (\text{HT\_Abs}) \qquad \Gamma \vdash V_2 : T'}{\Gamma \vdash (\lambda x.M_1)\ V_2 : \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}}\ (\text{HT\_App})$$

Therefore, $[\![M_1]\!]$ can take $\overline{v^{\mathsf{h}}}$. Then, the conclusion is shown as follows (here we choose the second disjunct of the conclusion and take $M' = M_1[\,V_2/x]$ as M''):

$$\begin{array}{rcll}
[\![M]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] & = & [\![(\lambda x.M_1)\ V_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] & \\
& = & [\![\lambda x.M_1]\!]\ [\![\,V_2]\!]\ \overline{v^{\mathsf{h}}}\ v^{\mathsf{k}} & \\
& = & (\lambda x.\mathsf{return}\ [\![M_1]\!])\ [\![\,V_2]\!]\ \overline{v^{\mathsf{h}}}\ v^{\mathsf{k}} & \\
& = & (\lambda x, \overline{h}, k.[\![M_1]\!][\,\overline{h} \mid k\,])\ [\![\,V_2]\!]\ \overline{v^{\mathsf{h}}}\ v^{\mathsf{k}} & \\
& \longrightarrow^{+} & [\![M_1]\!][\,\overline{h} \mid k\,][[\![\,V_2]\!]/x][\overline{v^{\mathsf{h}}}/\overline{h}][v^{\mathsf{k}}/k] & \\
& = & [\![M_1[\,V_2/x]]\!][\,\overline{h} \mid k\,][\overline{v^{\mathsf{h}}}/\overline{h}][v^{\mathsf{k}}/k] & \text{(by Lemma 12)} \\
& = & [\![M_1[\,V_2/x]]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] & \text{(by Lemma 13)}\ .
\end{array}$$

Case (HE_Fix): We are given

$$(\mathsf{fix}\, x.\, V_1)\ V_2 \longrightarrow V_1[\mathsf{fix}\, x.\, V_1/x]\ V_2$$

for some x, V_1, and V_2 such that $M = (\mathsf{fix}\, x.\, V_1)\ V_2$ and $M' = V_1[\mathsf{fix}\, x.\, V_1/x]\ V_2$. The conclusion is shown as follows (here we choose the second disjunct of the conclusion and take $M' = V_1[\mathsf{fix}\, x.\, V_1/x]\ V_2$ as M''):

$$\begin{array}{rcll}
[\![M]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] & = & [\![(\mathsf{fix}\, x.\, V_1)\ V_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] & \\
& = & [\![\mathsf{fix}\, x.\, V_1]\!]\ [\![\,V_2]\!]\ \overline{v^{\mathsf{h}}}\ v^{\mathsf{k}} & \\
& = & (\mathsf{fix}\, x.[\![\,V_1]\!])\ [\![\,V_2]\!]\ \overline{v^{\mathsf{h}}}\ v^{\mathsf{k}} & \\
& \longrightarrow & [\![\,V_1]\!][\mathsf{fix}\, x.[\![\,V_1]\!]/x]\ [\![\,V_2]\!]\ \overline{v^{\mathsf{h}}}\ v^{\mathsf{k}} & \\
& = & [\![\,V_1]\!][[\![\mathsf{fix}\, x.\, V_1]\!]/x]\ [\![\,V_2]\!]\ \overline{v^{\mathsf{h}}}\ v^{\mathsf{k}} & \\
& = & [\![\,V_1[\mathsf{fix}\, x.\, V_1/x]]\!]\ [\![\,V_2]\!]\ \overline{v^{\mathsf{h}}}\ v^{\mathsf{k}} & \text{(by Lemma 12)} \\
& = & [\![\,V_1[\mathsf{fix}\, x.\, V_1/x]\ V_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]\ . &
\end{array}$$

Case (HE_Case): We are given

$$\mathsf{case}(\underline{\mathsf{i}}; M_1, \cdots, M_n) \longrightarrow M_i$$

for some i, $M_1, \cdots, M_n$ such that $0 < i \leq n$, $M = \mathsf{case}(\underline{\mathsf{i}}; M_1, \cdots, M_n)$ and $M' = M_i$. The conclusion is shown as follows (here we choose the second disjunct of the conclusion and take $M' = M_i$ as M''):

$$\begin{array}{rcl}
[\![M]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] & = & [\![\mathsf{case}(\underline{\mathsf{i}}; M_1, \cdots, M_n)]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] \\
& = & \mathsf{case}([\![\underline{\mathsf{i}}]\!]; [\![M_1]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,], \cdots, [\![M_n]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]) \\
& = & \mathsf{case}(\underline{\mathsf{i}}; [\![M_1]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,], \cdots, [\![M_n]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]) \\
& \longrightarrow & [\![M_i]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]\ .
\end{array}$$

Case (HE_LetE): We are given

$$\frac{M_1 \longrightarrow M_1'}{\mathsf{let}\, x = M_1 \,\mathsf{in}\, M_2 \longrightarrow \mathsf{let}\, x = M_1' \,\mathsf{in}\, M_2}$$

for some x, M_1, M_2, and M_1' such that $M = (\mathsf{let}\, x = M_1 \,\mathsf{in}\, M_2)$ and $M' = (\mathsf{let}\, x = M_1' \,\mathsf{in}\, M_2)$. Because $\Gamma \vdash M : \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}$, we have the following derivation for some T_1 and A:

$$\frac{\Gamma \vdash M_1 : \Sigma \triangleright T_1 \,/\, A \Rightarrow A^{\mathrm{fin}} \qquad \Gamma, x : T_1 \vdash M_2 : \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A}{\Gamma \vdash \mathsf{let}\, x = M_1 \,\mathsf{in}\, M_2 : \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}} \;(\mathrm{HT\_Let})$$

By case analysis on the result of the IH on M_1.

Case $\forall \overline{v_0^{\mathsf{h}}}, v_0^{\mathsf{k}}.\ |\overline{v_0^{\mathsf{h}}}| = |\Sigma| \Longrightarrow \exists \sigma, V_0, y, M_0.\ M_1' \longrightarrow^* \sigma(V_0; y.\, M_0) \wedge [\![M_1]\!][\,\overline{v_0^{\mathsf{h}}} \mid v_0^{\mathsf{k}}\,] = [\![\sigma(V_0; y.\, M_0)]\!][\,\overline{v_0^{\mathsf{h}}} \mid v_0^{\mathsf{k}}\,]$: By the IH,

- $M_1' \longrightarrow^* \sigma(V_0; y.\, M_0)$ and
- $[\![M_1]\!][\,\overline{v^{\mathsf{h}}} \mid \lambda x.[\![M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]\,] = [\![\sigma(V_0; y.\, M_0)]\!][\,\overline{v^{\mathsf{h}}} \mid \lambda x.[\![M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]\,]$

for some V_0, y, and M_0. Without loss of generality, we can assume that $y \notin fv(M_2)$. By (HE_LetE) and (HE_LetOp),

$$M' = (\mathsf{let}\, x = M_1' \,\mathsf{in}\, M_2) \longrightarrow^* (\mathsf{let}\, x = \sigma(V_0; y.\, M_0) \,\mathsf{in}\, M_2) \longrightarrow \sigma(V_0; y.\, \mathsf{let}\, x = M_0 \,\mathsf{in}\, M_2)\ .$$

By Lemma 5, $\Gamma \vdash \sigma(V_0; y.\, M_0) : \Sigma \triangleright T_1 \,/\, A \Rightarrow A^{\mathrm{fin}}$. By its inversion, σ is included in Σ. Therefore, $\overline{v^{\mathsf{h}}}$ includes a value v_i^{h} corresponding to σ. Then, the conclusion is shown as follows (here we choose the first disjunct of the conclusion): if $\sigma_i : T_i^{\mathrm{par}} \rightsquigarrow T_i^{\mathrm{ari}} \,/\, C_i^{\mathrm{ini}} \Rightarrow A_i^{\mathrm{fin}} \in \Sigma$ for some T_i^{par}, T_i^{ari}, C_i^{ini}, and A_i^{fin}, then

$$\begin{array}{rcll} [\![M]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] & = & [\![\mathsf{let}\, x = M_1 \,\mathsf{in}\, M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] & \\ & = & [\![M_1]\!][\,\overline{v^{\mathsf{h}}} \mid \lambda x.[\![M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]\,] & \\ & = & [\![\sigma(V_0; y.\, M_0)]\!][\,\overline{v^{\mathsf{h}}} \mid \lambda x.[\![M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]\,] & \text{(by the IH)} \\ & = & v_i^{\mathsf{h}}\,[\![V_0]\!]\,\lambda y, \overline{h}, k.[\![M_0]\!][\,\overline{v^{\mathsf{h}}} \mid \lambda x.[\![M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]\,]\,\overline{h}\,k & \\ & = & v_i^{\mathsf{h}}\,[\![V_0]\!]\,\lambda y, \overline{h}, k.[\![\mathsf{let}\, x = M_0 \,\mathsf{in}\, M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]\,\overline{h}\,k & \\ & = & [\![\sigma(V_0; y.\, \mathsf{let}\, x = M_0 \,\mathsf{in}\, M_2)]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]\ ; & \end{array}$$

otherwise, if $\sigma_i : T_i^{\mathrm{par}} \rightsquigarrow T_i^{\mathrm{ari}} \,/\, T_i^{\mathrm{ini}} \Rightarrow A_i^{\mathrm{fin}} \in \Sigma$ for some T_i^{par}, T_i^{ari}, T_i^{ini}, and A_i^{fin}, then

$$\begin{array}{rcll} [\![M]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] & = & [\![\mathsf{let}\, x = M_1 \,\mathsf{in}\, M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] & \\ & = & [\![M_1]\!][\,\overline{v^{\mathsf{h}}} \mid \lambda x.[\![M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]\,] & \\ & = & [\![\sigma(V_0; y.\, M_0)]\!][\,\overline{v^{\mathsf{h}}} \mid \lambda x.[\![M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]\,] & \text{(by the IH)} \\ & = & v_i^{\mathsf{h}}\,[\![V_0]\!]\,\lambda y.[\![M_0]\!][\,\overline{v^{\mathsf{h}}} \mid \lambda x.[\![M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]\,] & \\ & = & v_i^{\mathsf{h}}\,[\![V_0]\!]\,\lambda y.[\![\mathsf{let}\, x = M_0 \,\mathsf{in}\, M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] & \\ & = & [\![\sigma(V_0; y.\, \mathsf{let}\, x = M_0 \,\mathsf{in}\, M_2)]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]\ . & \end{array}$$

Case $\forall \overline{v_0^{\mathsf{h}}}, v_0^{\mathsf{k}}.\ |\overline{v_0^{\mathsf{h}}}| = |\Sigma| \Longrightarrow \exists M_1''.\ M_1' \longrightarrow^* M_1'' \wedge [\![M_1]\!][\,\overline{v_0^{\mathsf{h}}} \mid v_0^{\mathsf{k}}\,] \longrightarrow^+ [\![M_1'']\!][\,\overline{v_0^{\mathsf{h}}} \mid v_0^{\mathsf{k}}\,]$: By the IH,

- $M_1' \longrightarrow^* M_1''$ and
- $[\![M_1]\!][\,\overline{v^{\mathsf{h}}} \mid \lambda x.[\![M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]\,] \longrightarrow^+ [\![M_1'']\!][\,\overline{v^{\mathsf{h}}} \mid \lambda x.[\![M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]\,]$

for some M_1''. By (HE_LetE),

$$M' = (\mathsf{let}\, x = M_1' \,\mathsf{in}\, M_2) \longrightarrow^* (\mathsf{let}\, x = M_1'' \,\mathsf{in}\, M_2)\ .$$

Therefore, the conclusion is shown as follows (here we choose the second disjunct of the conclusion and take $\mathsf{let}\, x = M_1'' \,\mathsf{in}\, M_2$ as M''):

$$\begin{array}{rcll} [\![M]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] & = & [\![\mathsf{let}\, x = M_1 \,\mathsf{in}\, M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] & \\ & = & [\![M_1]\!][\,\overline{v^{\mathsf{h}}} \mid \lambda x.[\![M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]\,] & \\ & \longrightarrow^+ & [\![M_1'']\!][\,\overline{v^{\mathsf{h}}} \mid \lambda x.[\![M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]\,] & \text{(by the IH)} \\ & = & [\![\mathsf{let}\, x = M_1'' \,\mathsf{in}\, M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]\ . & \end{array}$$

Case (HE_LetV): We are given

$$\mathsf{let}\,x = \mathsf{return}\ V_1\ \mathsf{in}\ M_2 \longrightarrow M_2[V_1/x]$$

for some x, V_1, and M_2 such that $M = (\mathsf{let}\,x = \mathsf{return}\ V_1\ \mathsf{in}\ M_2)$ and $M' = M_2[V_1/x]$. Without loss of generality, we can assume that $x \notin fv(\overline{v^{\mathsf{h}}}) \cup fv(v^{\mathsf{k}})$. Then, the conclusion is shown as follows (here we choose the second disjunct of the conclusion and take $M' = M_2[V_1/x]$ as M''):

$$\begin{array}{rcl} [\![M]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] & = & [\![\mathsf{let}\,x = \mathsf{return}\ V_1\ \mathsf{in}\ M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] \\ & = & [\![\mathsf{return}\ V_1]\!][\,\overline{v^{\mathsf{h}}} \mid \lambda x.[\![M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]\,] \\ & = & (\lambda x.[\![M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,])\,[\![V_1]\!] \\ & \longrightarrow & [\![M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,][[\![V_1]\!]/x] \\ & = & [\![M_2[V_1/x]]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] \quad \text{(by Lemma 12; note that } x \text{ does not occur free in } \overline{v^{\mathsf{h}}} \text{ and } v^{\mathsf{k}}) . \end{array}$$

Case (HE_LetOp): We are given

$$\mathsf{let}\,x = \sigma(V_1; y.\,M_1)\ \mathsf{in}\ M_2 \longrightarrow \sigma(V_1; y.\,\mathsf{let}\,x = M_1\ \mathsf{in}\ M_2)$$

for some x, y, σ, V_1, M_1, and M_2 such that $y \notin fv(M_2)$ and $M = (\mathsf{let}\,x = \sigma(V_1; y.\,M_1)\ \mathsf{in}\ M_2)$ and $M' = \sigma(V_1; y.\,\mathsf{let}\,x = M_1\ \mathsf{in}\ M_2)$. Without loss of generality, we can assume that $y \notin fv(\overline{v^{\mathsf{h}}}) \cup fv(v^{\mathsf{k}})$. Because $\Gamma \vdash M : \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}$, its inversion implies that σ is included in Σ. Therefore, $\overline{v^{\mathsf{h}}}$ includes a value v_i^{h} corresponding to σ. Then, the conclusion is shown as follows (here we choose the first disjunct of the conclusion): if $\sigma : T_i^{\mathrm{par}} \rightsquigarrow T_i^{\mathrm{ari}} \,/\, C_i^{\mathrm{ini}} \Rightarrow A_i^{\mathrm{fin}} \in \Sigma$ for some T_i^{par}, T_i^{ari}, C_i^{ini}, and A_i^{fin}, then

$$\begin{array}{rcl} [\![M]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] & = & [\![\mathsf{let}\,x = \sigma(V_1; y.\,M_1)\ \mathsf{in}\ M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] \\ & = & [\![\sigma(V_1; y.\,M_1)]\!][\,\overline{v^{\mathsf{h}}} \mid \lambda x.[\![M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]\,] \\ & = & v_i^{\mathsf{h}}\,[\![V_1]\!]\,(\lambda y, \overline{h}, k.[\![M_1]\!][\,\overline{v^{\mathsf{h}}} \mid \lambda x.[\![M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]\,]\,\overline{h}\,k) \\ & = & v_i^{\mathsf{h}}\,[\![V_1]\!]\,(\lambda y, \overline{h}, k.[\![\mathsf{let}\,x = M_1\ \mathsf{in}\ M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]\,\overline{h}\,k) \\ & = & [\![\sigma(V_1; y.\,\mathsf{let}\,x = M_1\ \mathsf{in}\ M_2)]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] \;; \end{array}$$

otherwise, if $\sigma : T_i^{\mathrm{par}} \rightsquigarrow T_i^{\mathrm{ari}} \,/\, T_i^{\mathrm{ini}} \Rightarrow A_i^{\mathrm{fin}} \in \Sigma$ for some T_i^{par}, T_i^{ari}, T_i^{ini}, and A_i^{fin}, then

$$\begin{array}{rcl} [\![M]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] & = & [\![\mathsf{let}\,x = \sigma(V_1; y.\,M_1)\ \mathsf{in}\ M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] \\ & = & [\![\sigma(V_1; y.\,M_1)]\!][\,\overline{v^{\mathsf{h}}} \mid \lambda x.[\![M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]\,] \\ & = & v_i^{\mathsf{h}}\,[\![V_1]\!]\,(\lambda y.[\![M_1]\!][\,\overline{v^{\mathsf{h}}} \mid \lambda x.[\![M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]\,]) \\ & = & v_i^{\mathsf{h}}\,[\![V_1]\!]\,(\lambda y.[\![\mathsf{let}\,x = M_1\ \mathsf{in}\ M_2]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]) \\ & = & [\![\sigma(V_1; y.\,\mathsf{let}\,x = M_1\ \mathsf{in}\ M_2)]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] \;. \end{array}$$

Note that here $\sigma(V_1; y.\,\mathsf{let}\,x = M_1\ \mathsf{in}\ M_2)$ can be typed at the operation signature Σ by Lemma 5.

Case (HE_HandleE): We are given

$$\frac{M_0 \longrightarrow M_0'}{\mathsf{with}\ H\ \mathsf{handle}\ M_0 \longrightarrow \mathsf{with}\ H\ \mathsf{handle}\ M_0'}$$

for some H, M_0, and M_0' such that $M = \mathsf{with}\ H\ \mathsf{handle}\ M_0$ and $M' = \mathsf{with}\ H\ \mathsf{handle}\ M_0'$. Let $H = \{\mathsf{return}\,x \mapsto M_0''\} \uplus \{\sigma_i(x_i; k_i) \mapsto M_i\}^{1 \le i \le n}$ for some x, M_0'', $x_1, \cdots, x_n$, $k_1, \cdots, k_n$, $\sigma_1, \cdots, \sigma_n$ $M_1, \cdots, M_n$. Because $\Gamma \vdash M : \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}$, we have the following derivation for some $T_1^{\mathrm{par}}, \cdots, T_n^{\mathrm{par}}$, $T_1^{\mathrm{ari}}, \cdots, T_n^{\mathrm{ari}}$, $C_1^{\mathrm{ini}}, \cdots, C_n^{\mathrm{ini}}$, $C_1^{\mathrm{fin}}, \cdots, C_n^{\mathrm{fin}}$, Σ_0, T_0, C_0^{ini}, and C_0^{fin}:

$$\frac{\begin{array}{c} \Sigma_0 = \{\sigma_i : T_i^{\mathrm{par}} \rightsquigarrow T_i^{\mathrm{ari}} \,/\, C_i^{\mathrm{ini}} \Rightarrow C_i^{\mathrm{fin}}\}^{1 \le i \le n} \qquad C_0^{\mathrm{fin}} = \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}} \\ \Gamma \vdash M_0 : \Sigma_0 \triangleright T_0 \,/\, C_0^{\mathrm{ini}} \Rightarrow C_0^{\mathrm{fin}} \\ \Gamma, x : T_0 \vdash M_0'' : C_0^{\mathrm{ini}} \qquad \forall i \in [1,n].\ \Gamma, x_i : T_i^{\mathrm{par}}, k_i : T_i^{\mathrm{ari}} \to C_i^{\mathrm{ini}} \vdash M_i : C_i^{\mathrm{fin}} \end{array}}{\Gamma \vdash \mathsf{with}\ H\ \mathsf{handle}\ M_0 : \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}}\ (\text{HT\_Handle})$$

Therefore, we can apply the IH on M_0. By case analysis on its result. In what follows, let $v_i = \lambda x_i, k_i.\mathsf{return}\,[\![M_i]\!]$ for any $i \in [1,n]$.

Case $\forall\, \overline{v_0^{\mathsf{h}}}, v_0^{\mathsf{k}}.\ |\overline{v_0^{\mathsf{h}}}| = |\Sigma_0| \Longrightarrow \exists\, \sigma, V_0, y, M_0'''.\ M_0' \longrightarrow^* \sigma(V_0; y.\, M_0''') \ \wedge\ [\![M_0]\!][\,\overline{v_0^{\mathsf{h}}} \mid v_0^{\mathsf{k}}\,] = [\![\sigma(V_0; y.\, M_0''')]\!][\,\overline{v_0^{\mathsf{h}}} \mid v_0^{\mathsf{k}}\,]$: By the IH,

- $M_0' \longrightarrow^* \sigma(V_0; y.\, M_0''')$ and
- $[\![M_0]\!][\,v_1, \cdots, v_n \mid \lambda x.\mathsf{return}\,[\![M_0'']\!]\,] = [\![\sigma(V_0; y.\, M_0''')]\!][\,v_1, \cdots, v_n \mid \lambda x.\mathsf{return}\,[\![M_0'']\!]\,]$

for some σ, V_0, y, and M_0'''. By (HE_HandleE),

$$M' = \mathsf{with}\ H\ \mathsf{handle}\ M_0' \longrightarrow^* \mathsf{with}\ H\ \mathsf{handle}\ \sigma(V_0; y.\, M_0''')\ .$$

By Lemma 5 and the inversion of the typing derivation, $\sigma = \sigma_i$ for some i. By (HE_HandleOp),

$$M' \longrightarrow^* \mathsf{with}\ H\ \mathsf{handle}\ \sigma(V_0; y.\, M_0''') \longrightarrow M_i[\,V_0/x_i][\lambda y.\mathsf{with}\ H\ \mathsf{handle}\ M_0'''/k_i]\ .$$

Then, the conclusion is shown as follows (here we choose the second disjunct of the conclusion):

$$\begin{array}{rcll}
[\![M]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] & = & [\![\mathsf{with}\ H\ \mathsf{handle}\ M_0]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] & \\
& = & [\![M_0]\!][\,v_1, \cdots, v_n \mid \lambda x.\mathsf{return}\,[\![M_0'']\!]\,]\,\overline{v^{\mathsf{h}}}\,v^{\mathsf{k}} & \\
& = & [\![\sigma(V_0; y.\, M_0''')]\!][\,v_1, \cdots, v_n \mid \lambda x.\mathsf{return}\,[\![M_0'']\!]\,]\,\overline{v^{\mathsf{h}}}\,v^{\mathsf{k}} & \text{(by the IH)} \\
& = & v_i\,[\![V_0]\!]\,(\lambda y, \overline{h}, k.[\![M_0''']\!][\,v_1, \cdots, v_n \mid \lambda x.\mathsf{return}\,[\![M_0'']\!]\,]\,\overline{h}\,k)\,\overline{v^{\mathsf{h}}}\,v^{\mathsf{k}} & \\
& = & v_i\,[\![V_0]\!]\,(\lambda y, \overline{h}, k.[\![\mathsf{with}\ H\ \mathsf{handle}\ M_0''']\!][\,\overline{h} \mid k\,])\,\overline{v^{\mathsf{h}}}\,v^{\mathsf{k}} & \\
& = & v_i\,[\![V_0]\!]\,[\![\lambda y.\mathsf{with}\ H\ \mathsf{handle}\ M_0''']\!]\,\overline{v^{\mathsf{h}}}\,v^{\mathsf{k}} & \\
& \longrightarrow^+ & \mathsf{return}\,[\![M_i]\!][[\![V_0]\!]/x_i][[\![\lambda y.\mathsf{with}\ H\ \mathsf{handle}\ M_0''']\!]/k_i]\,\overline{v^{\mathsf{h}}}\,v^{\mathsf{k}} & \\
& = & (\lambda \overline{h}, k.[\![M_i[\,V_0/x_i][\lambda y.\mathsf{with}\ H\ \mathsf{handle}\ M_0'''/k_i]]\!][\,\overline{h} \mid k\,])\,\overline{v^{\mathsf{h}}}\,v^{\mathsf{k}} & \text{by Lemma 12} \\
& \longrightarrow^+ & [\![M_i[\,V_0/x_i][\lambda y.\mathsf{with}\ H\ \mathsf{handle}\ M_0'''/k_i]]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] & \text{by Lemma 13}\ .
\end{array}$$

Note the following points.

- The term $\sigma(V_0; y.\, M_0''')$ can be typed at the operation signature Σ_0 by Lemma 5, and Σ_0 assigns to σ the type $T_i^{\mathrm{par}} \to T_i^{\mathrm{ari}}\ /\ C_i^{\mathrm{ini}} \Rightarrow C_i^{\mathrm{fin}}$; thus, $[\![\sigma(V_0; y.\, M_0''')]\!][\,v_1, \cdots, v_n \mid \lambda x.\mathsf{return}\,[\![M_0'']\!]\,]$ involves the term
$$\lambda y, \overline{h}, k.[\![M_0''']\!][\,v_1, \cdots, v_n \mid \lambda x.\mathsf{return}\,[\![M_0'']\!]\,]\,\overline{h}\,k$$
in the eta-expanded form.
- Because $\Gamma \vdash \sigma(V_0; y.\, M_0''') : \Sigma_0 \triangleright T_0\ /\ C_0^{\mathrm{ini}} \Rightarrow C_0^{\mathrm{fin}}$ by Lemma 5, its inversion implies $C_i^{\mathrm{fin}} = C_0^{\mathrm{fin}} = \Sigma \triangleright T\ /\ A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}$. Because $\Gamma, x_i : T_i^{\mathrm{par}}, k_i : T_i^{\mathrm{ari}} \to C_i^{\mathrm{ini}} \vdash M_i : C_i^{\mathrm{fin}}$, $[\![M_i]\!] = \lambda \overline{h}, k.[\![M_i]\!][\,\overline{h} \mid k\,]$ can take $\overline{v^{\mathsf{h}}}$.

Case $\forall\, \overline{v_0^{\mathsf{h}}}, v_0^{\mathsf{k}}.\ |\overline{v_0^{\mathsf{h}}}| = |\Sigma_0| \Longrightarrow \exists\, M_0'''.\ M_0' \longrightarrow^* M_0''' \ \wedge\ [\![M_0]\!][\,\overline{v_0^{\mathsf{h}}} \mid v_0^{\mathsf{k}}\,] \longrightarrow^+ [\![M_0''']\!][\,\overline{v_0^{\mathsf{h}}} \mid v_0^{\mathsf{k}}\,]$: By the IH,

- $M_0' \longrightarrow^* M_0'''$ and
- $[\![M_0]\!][\,v_1, \cdots, v_n \mid \lambda x.\mathsf{return}\,[\![M_0'']\!]\,] \longrightarrow^+ [\![M_0''']\!][\,v_1, \cdots, v_n \mid \lambda x.\mathsf{return}\,[\![M_0'']\!]\,]$

for some M_0'''. By (HE_HandleE),

$$M' = \mathsf{with}\ H\ \mathsf{handle}\ M_0' \longrightarrow^* \mathsf{with}\ H\ \mathsf{handle}\ M_0'''\ .$$

Therefore, the conclusion is shown as follows (here we choose the second disjunct of the conclusion and take $\mathsf{with}\ H\ \mathsf{handle}\ M_0'''$ as M''):

$$\begin{array}{rcll}
[\![M]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] & = & [\![\mathsf{with}\ H\ \mathsf{handle}\ M_0]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] & \\
& = & [\![M_0]\!][\,v_1, \cdots, v_n \mid \lambda x.\mathsf{return}\,[\![M_0'']\!]\,]\,\overline{v^{\mathsf{h}}}\,v^{\mathsf{k}} & \\
& \longrightarrow^+ & [\![M_0''']\!][\,v_1, \cdots, v_n \mid \lambda x.\mathsf{return}\,[\![M_0'']\!]\,]\,\overline{v^{\mathsf{h}}}\,v^{\mathsf{k}} & \text{(by the IH)} \\
& = & [\![\mathsf{with}\ H\ \mathsf{handle}\ M_0''']\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]\ . &
\end{array}$$

Case (HE_HandleV): We are given

$$\mathsf{with}\ H\ \mathsf{handle}\ \mathsf{return}\ V \longrightarrow M_0[\,V/x]$$

for some H, V, M_0, and x such that $\mathsf{return}\, x \mapsto M_0 \in H$ and $M = \mathsf{with}\, H\, \mathsf{handle}\, \mathsf{return}\, V$ and $M' = M_0[V/x]$. Without loss of generality, we can assume that $x \notin fv(\overline{v^\mathsf{h}}) \cup fv(v^\mathsf{k})$. Let $H = \{\mathsf{return}\, x \mapsto M_0\} \uplus \{\sigma_i(x_i; k_i) \mapsto M_i\}^{1 \le i \le n}$ for some $x_1, \cdots, x_n$, $k_1, \cdots, k_n$, $\sigma_1, \cdots, \sigma_n$ $M_1, \cdots, M_n$. Furthermore, let $v_i = \lambda x_i, k_i.\mathsf{return}\, [\![M_i]\!]$ for any $i \in [1, n]$. Because $\Gamma \vdash M : \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}$, we have the following derivation for some $T_1^{\mathrm{par}}, \cdots, T_n^{\mathrm{par}}$, $T_1^{\mathrm{ari}}, \cdots, T_n^{\mathrm{ari}}$, $C_1^{\mathrm{ini}}, \cdots, C_n^{\mathrm{ini}}$, $C_1^{\mathrm{fin}}, \cdots, C_n^{\mathrm{fin}}$, Σ_0, T_0, C_0^{ini}, and C_0^{fin}:

$$\frac{\begin{array}{c}\Sigma_0 = \{\sigma_i : T_i^{\mathrm{par}} \rightsquigarrow T_i^{\mathrm{ari}} \,/\, C_i^{\mathrm{ini}} \Rightarrow C_i^{\mathrm{fin}}\}^{1 \le i \le n} \qquad C_0^{\mathrm{fin}} = \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}} \\ \Gamma \vdash \mathsf{return}\, V : \Sigma_0 \triangleright T_0 \,/\, C_0^{\mathrm{ini}} \Rightarrow C_0^{\mathrm{fin}} \\ \Gamma, x : T_0 \vdash M_0 : C_0^{\mathrm{ini}} \qquad \forall\, i \in [1, n].\ \Gamma, x_i : T_i^{\mathrm{par}}, k_i : T_i^{\mathrm{ari}} \to C_i^{\mathrm{ini}} \vdash M_i : C_i^{\mathrm{fin}}\end{array}}{\Gamma \vdash \mathsf{with}\, H\, \mathsf{handle}\, \mathsf{return}\, V : \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}}\ (\text{HT\_Handle})$$

By inversion of $\Gamma \vdash \mathsf{return}\, V : \Sigma_0 \triangleright T_0 \,/\, C_0^{\mathrm{ini}} \Rightarrow C_0^{\mathrm{fin}}$, we have $C_0^{\mathrm{ini}} = C_0^{\mathrm{fin}}$, that is, $C_0^{\mathrm{ini}} = \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}$. Because $\Gamma, x : T_0 \vdash M_0 : C_0^{\mathrm{ini}}$, we can find that $[\![M_0]\!]$ can take $\overline{v^\mathsf{h}}$. Then, the conclusion is shown as follows (here we choose the second disjunct of the conclusion and take $M' = M_0[V/x]$ as M''):

$$\begin{array}{rcll}
[\![M]\!][\,\overline{v^\mathsf{h}} \mid v^\mathsf{k}\,] & = & [\![\mathsf{with}\, H\, \mathsf{handle}\, \mathsf{return}\ V]\!][\,\overline{v^\mathsf{h}} \mid v^\mathsf{k}\,] & \\
& = & [\![\mathsf{return}\ V]\!][\,v_1, \cdots, v_n \mid \lambda x.\mathsf{return}\, [\![M_0]\!]\,]\, \overline{v^\mathsf{h}}\, v^\mathsf{k} & \\
& = & (\lambda x.\mathsf{return}\, [\![M_0]\!])\, [\![V]\!]\, \overline{v^\mathsf{h}}\, v^\mathsf{k} & \\
& = & (\lambda x, \overline{h}, k.[\![M_0]\!][\,\overline{h} \mid k\,])\, [\![V]\!]\, \overline{v^\mathsf{h}}\, v^\mathsf{k} & \\
& \longrightarrow^+ & [\![M_0]\!][\,\overline{h} \mid k\,][[\![V]\!]/x][\overline{v^\mathsf{h}}/\overline{h}][v^\mathsf{k}/k] & \\
& = & [\![M_0[V/x]]\!][\,\overline{v^\mathsf{h}} \mid v^\mathsf{k}\,] & \text{by Lemmas 12 and 13}\ .
\end{array}$$

Case (HE_HandleOp): We are given

$$\mathsf{with}\, H\, \mathsf{handle}\, \sigma(V; y.\, M_0') \longrightarrow M_0''[V/x''][\lambda y.\mathsf{with}\, H\, \mathsf{handle}\, M_0'/k'']$$

for some H, V, y, M_0', M_0'', x'', and k'' such that $\sigma(x''; k'') \mapsto M_0'' \in H$ and $M = \mathsf{with}\, H\, \mathsf{handle}\, \sigma(V; y.\, M_0')$ and $M' = M_0''[V/x''][\lambda y.\mathsf{with}\, H\, \mathsf{handle}\, M_0'/k'']$. Without loss of generality, we can assume that $x'', k'' \notin fv(\overline{v^\mathsf{h}}) \cup fv(v^\mathsf{k})$. Let $H = \{\mathsf{return}\, x \mapsto M_0\} \uplus \{\sigma_i(x_i; k_i) \mapsto M_i\}^{1 \le i \le n}$ for some x, M_0, $x_1, \cdots, x_n$, $k_1, \cdots, k_n$, $\sigma_1, \cdots, \sigma_n$ $M_1, \cdots, M_n$. We have some $j > 0$ such that $\sigma = \sigma_j$, $x'' = x_j$, $k'' = k_j$, and $M_0'' = M_j$. Furthermore, let $v_i = \lambda x_i, k_i.\mathsf{return}\, [\![M_i]\!]$ for any $i \in [1, n]$. Because $\Gamma \vdash M : \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}$, we have the following derivation for some $T_1^{\mathrm{par}}, \cdots, T_n^{\mathrm{par}}$, $T_1^{\mathrm{ari}}, \cdots, T_n^{\mathrm{ari}}$, $C_1^{\mathrm{ini}}, \cdots, C_n^{\mathrm{ini}}$, $C_1^{\mathrm{fin}}, \cdots, C_n^{\mathrm{fin}}$, Σ_0, T_0, C_0^{ini}, and C_0^{fin}:

$$\frac{\begin{array}{c}\Sigma_0 = \{\sigma_i : T_i^{\mathrm{par}} \rightsquigarrow T_i^{\mathrm{ari}} \,/\, C_i^{\mathrm{ini}} \Rightarrow C_i^{\mathrm{fin}}\}^{1 \le i \le n} \qquad C_0^{\mathrm{fin}} = \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}} \\ \Gamma \vdash \sigma(V; y.\, M_0') : \Sigma_0 \triangleright T_0 \,/\, C_0^{\mathrm{ini}} \Rightarrow C_0^{\mathrm{fin}} \\ \Gamma, x : T_0 \vdash M_0 : C_0^{\mathrm{ini}} \qquad \forall\, i \in [1, n].\ \Gamma, x_i : T_i^{\mathrm{par}}, k_i : T_i^{\mathrm{ari}} \to C_i^{\mathrm{ini}} \vdash M_i : C_i^{\mathrm{fin}}\end{array}}{\Gamma \vdash \mathsf{with}\, H\, \mathsf{handle}\, \sigma(V; y.\, M_0') : \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}}\ (\text{HT\_Handle})$$

By inversion of $\Gamma \vdash \sigma(V; y.\, M_0') : \Sigma_0 \triangleright T_0 \,/\, C_0^{\mathrm{ini}} \Rightarrow C_0^{\mathrm{fin}}$ and $\sigma = \sigma_j$, we have $C_0^{\mathrm{fin}} = C_j^{\mathrm{fin}}$, that is, $C_j^{\mathrm{fin}} = \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}$. Because $\Gamma, x_j : T_j^{\mathrm{par}}, k : T_j^{\mathrm{ari}} \to C_j^{\mathrm{ini}} \vdash M_j : C_j^{\mathrm{fin}}$, we can find that $[\![M_j]\!]$ can take $\overline{v^\mathsf{h}}$. Then, the conclusion is shown as follows (here we choose the second disjunct of the conclusion and take $M' = M_j[V/x_j][\lambda y.\mathsf{with}\, H\, \mathsf{handle}\, M_0'/k_j]$ as M''):

$$\begin{array}{rcll}
[\![M]\!][\,\overline{v^\mathsf{h}} \mid v^\mathsf{k}\,] & = & [\![\mathsf{with}\, H\, \mathsf{handle}\, \sigma(V; y.\, M_0')]\!][\,\overline{v^\mathsf{h}} \mid v^\mathsf{k}\,] & \\
& = & [\![\sigma_j(V; y.\, M_0')]\!][\,v_1, \cdots, v_n \mid \lambda x.\mathsf{return}\, [\![M_0]\!]\,]\, \overline{v^\mathsf{h}}\, v^\mathsf{k} & \\
& = & v_j\, [\![V]\!]\, (\lambda y, \overline{h_0}, k_0.[\![M_0']\!][\,v_1, \cdots, v_n \mid \lambda x.\mathsf{return}\, [\![M_0]\!]\,]\, \overline{h_0}\, k_0)\, \overline{v^\mathsf{h}}\, v^\mathsf{k} & \\
& = & v_j\, [\![V]\!]\, (\lambda y, \overline{h_0}, k_0.[\![\mathsf{with}\, H\, \mathsf{handle}\, M_0']\!][\,\overline{h_0} \mid k_0\,])\, \overline{v^\mathsf{h}}\, v^\mathsf{k} & \\
& = & v_j\, [\![V]\!]\, [\![\lambda y.\mathsf{with}\, H\, \mathsf{handle}\, M_0']\!]\, \overline{v^\mathsf{h}}\, v^\mathsf{k} & \\
& = & (\lambda x_j, k_j, \overline{h}, k.[\![M_j]\!][\,\overline{h} \mid k\,])\, [\![V]\!]\, [\![\lambda y.\mathsf{with}\, H\, \mathsf{handle}\, M_0']\!]\, \overline{v^\mathsf{h}}\, v^\mathsf{k} & \\
& \longrightarrow^+ & [\![M_j]\!][\,\overline{h} \mid k\,][[\![V]\!]/x_j][[\![\lambda y.\mathsf{with}\, H\, \mathsf{handle}\, M_0']\!]/k_j][\overline{v^\mathsf{h}}/\overline{h}][v^\mathsf{k}/k] & \\
& = & [\![M_j[V/x_j][\lambda y.\mathsf{with}\, H\, \mathsf{handle}\, M_0'/k_j]]\!][\,\overline{v^\mathsf{h}} \mid v^\mathsf{k}\,] & \text{(by Lemmas 12 and 13)}\ .
\end{array}$$

Note that the term $\sigma(V; y.\, M'_0)$ is typed at the operation signature Σ_0, which assigns to σ_j the type $T_j^{\mathrm{par}} \to T_j^{\mathrm{ari}} \,/\, C_j^{\mathrm{ini}} \Rightarrow C_j^{\mathrm{fin}}$. Thus, $[\![\sigma_i(V; y.\, M'_0)]\!][\, v_1, \cdots, v_n \mid \lambda x.\mathsf{return}\; [\![M_0]\!]\,]$ involves the term

$$\lambda y, \overline{h_0}, k_0.[\![M'_0]\!][\, v_1, \cdots, v_n \mid \lambda x.\mathsf{return}\; [\![M_0]\!]\,]\; \overline{h_0}\; k_0$$

in the eta-expanded form.

□

Lemma 15 (Evaluation in $\mathsf{HEPCF}^{\mathsf{ATM}}$ is Deterministic)**.** *If $M \longrightarrow M_1$ and $M \longrightarrow M_2$, then $M_1 = M_2$.*

Proof. Straightforward by induction on the derivation of $M \longrightarrow M_1$. □

Lemma 16 (Well-Definedness of $\mathsf{HEPCF}^{\mathsf{ATM}}$ Effect Trees)**.** *If $\emptyset \vdash M : \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}$ and Σ is top-level, then $\mathbf{ET}(M)$ is well defined and uniquely determined, and it is in $\mathbf{Tree}_{S_T^\Sigma}$.*

Proof. We show that $\mathbf{ET}(M) \in \mathbf{Tree}_{S_T^\Sigma}$ by coinduction. We proceed by case analysis on the evaluation of M.

Case $M \longrightarrow^\omega$: Obvious.

Case $\exists\, V.\ M \longrightarrow^* \mathsf{return}\; V$: By the definition, $\mathbf{ET}(M) = \mathsf{return}\; V$. By Lemma 5, $\emptyset \vdash \mathsf{return}\; V : \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}$. By its inversion, $\emptyset \vdash V : T$. Thus, $\mathsf{return}\; V \in \mathbf{Tree}_{S_T^\Sigma}$.

Case $\exists\, \sigma, V, x, M'.\ M \longrightarrow^* \sigma(V; x.\, M')$: By Lemma 5, $\emptyset \vdash \sigma(V; x.\, M') : \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}$. By its inversion and Lemma 3,

- $\sigma : B \rightsquigarrow \mathsf{n} \,/\, T' \Rightarrow T' \in \Sigma$,
- $V = c$, and
- $x : \mathsf{n} \vdash M' : \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow T'$

for some B, n, T', and c (note that Σ is top-level). Then, by the definition,

$$\mathbf{ET}(M) = \sigma(c, \mathbf{ET}(M'[\underline{1}/x]), \cdots, \mathbf{ET}(M'[\underline{\mathsf{n}}/x]))\ .$$

Thus, by the coinduction principle, it suffices to show that, for any $i \in [1, n]$, $\emptyset \vdash M'[\underline{\mathsf{i}}/x] : \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow T'$, which is shown by Lemma 2 with $x : \mathsf{n} \vdash M' : \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow T'$ and $\emptyset \vdash \underline{\mathsf{i}} : \mathsf{n}$.

Otherwise: Contradictory with Lemmas 5 and 4.

The uniqueness of $\mathbf{ET}(M)$ is shown by Lemma 15. □

Lemma 17 (Evaluation in EPCF is Deterministic)**.** *If $e \longrightarrow e_1$ and $e \longrightarrow e_2$, then $e_1 = e_2$.*

Proof. Straightforward by induction on the derivation of $e \longrightarrow e_1$. □

Lemma 18 (Well-Definedness of EPCF Effect Trees)**.** *If $\Xi \parallel \emptyset \vdash e : \tau$, then $\mathbf{ET}(e)$ is well defined and uniquely determined, and it is in $\mathbf{Tree}_{S_\tau^\Xi}$.*

Proof. We show that $\mathbf{ET}(e) \in \mathbf{Tree}_{S_\tau^\Xi}$ by coinduction. We proceed by case analysis on the evaluation of e.

Case $e \longrightarrow^\omega$: Obvious.

Case $\exists\, v.\ e \longrightarrow^* \mathsf{return}\; v$: By the definition, $\mathbf{ET}(e) = \mathsf{return}\; v$. By Lemma 10, $\Xi \parallel \emptyset \vdash \mathsf{return}\; v : \tau$. By its inversion, $\Xi \parallel \emptyset \vdash v : \tau$. Thus, $\mathsf{return}\; v \in \mathbf{Tree}_{S_\tau^\Xi}$.

Case $\exists\, \sigma, v, x, e'.\ e \longrightarrow^* \sigma(v; x.\, e')$: By Lemma 10, $\Xi \parallel \emptyset \vdash \sigma(v; x.\, e') : \tau$. By its inversion and Lemma 8,

- $\sigma : B \rightsquigarrow \mathsf{n} \in \Xi$,
- $v = c$, and
- $\Xi \parallel x : \mathsf{n} \vdash e' : \tau$

for some B n, and c. Then, by the definition, $\mathbf{ET}(e) = \sigma(c, \mathbf{ET}(e'[\underline{1}/x]), \cdots, \mathbf{ET}(e'[\underline{n}/x]))$. Thus, by the coinduction principle, it suffices to show that, for any $i \in [1, n]$, $\Xi \parallel \emptyset \vdash e'[\underline{i}/x] : \tau$, which is shown by Lemma 7 with $\Xi \parallel x : \mathsf{n} \vdash e' : \tau$ and $\Xi \parallel \emptyset \vdash \underline{i} : \mathsf{n}$.

Otherwise: Contradictory with Lemmas 10 and 9.

The uniqueness of $\mathbf{ET}(e)$ is shown by Lemma 17. □

Lemma 19 (Evaluation Preserves Effect Trees in EPCF)**.** *If* $\Xi \parallel \emptyset \vdash e : \tau$ *and* $e \longrightarrow^* e'$, *then* $\mathbf{ET}(e) = \mathbf{ET}(e')$.

Proof. By Lemmas 10 and 18, $\mathbf{ET}(e), \mathbf{ET}(e') \in \mathbf{Tree}_{S^{\Xi}_{\tau}}$. We show that $\mathbf{ET}(e) = \mathbf{ET}(e')$ by case analysis on the evaluation of e.

Case $e \longrightarrow^{\omega}$: By Lemma 17, $e' \longrightarrow^{\omega}$. Therefore, $\mathbf{ET}(e) = \mathbf{ET}(e') = \bot$.

Case $\exists\, v.\ e \longrightarrow^* \mathsf{return}\, v$: By Lemma 17, $e' \longrightarrow^* \mathsf{return}\, v$. Therefore, $\mathbf{ET}(e) = \mathbf{ET}(e') = \mathsf{return}\, v$.

Case $\exists\, \sigma, v, x, e_0.\ e \longrightarrow^* \sigma(v; x.\, e_0)$: Because $\mathbf{ET}(e)$ is well defined, we have $\sigma : B \rightsquigarrow \mathsf{n} \in \Xi$ and $v = c$ for some B, $\underline{n}$, and c. By Lemma 17, $e' \longrightarrow^* \sigma(c; x.\, e_0)$. Therefore, $\mathbf{ET}(e) = \mathbf{ET}(e') = \sigma(c, \mathbf{ET}(e_0[\underline{1}/x]), \cdots, \mathbf{ET}(e_0[\underline{n}/x]))$.

Otherwise: Contradictory with Lemmas 10 and 9.

□

Lemma 20 (Correspondence between Effect Trees of CPS-Transformed Terms and CPS-Transformed Effect Trees)**.** *Let* $\Sigma = \{\sigma_i : B_i \rightsquigarrow E_i \,/\, T_i \Rightarrow \underline{T_i}\}^{1 \le i \le n}$ *and* $\Xi = \{\sigma_i : B_i \rightsquigarrow E_i\}^{1 \le i \le n}$. *Assume that* $\emptyset \vdash M : \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}$ *and* $\sigma_1, \cdots, \sigma_n$ *are ordered. Let* $\overline{v^{\mathsf{h}}} = v^{\mathsf{h}}_1, \cdots, v^{\mathsf{h}}_n$ *such that, for any* $i \in [1, n]$, $v^{\mathsf{h}}_i = \lambda x, k.\sigma_i(x; y.\, k\, y)$ *for some distinct variables* x, k, *and* y. *Also, let* v^{k} *be a value such that* $\Xi \parallel \emptyset \vdash v^{\mathsf{k}} : [\![T]\!] \to [\![A^{\mathrm{ini}}]\!]$. *Then,* $\mathbf{ET}([\![M]\!][\,\overline{v^{\mathsf{h}}} \,|\, v^{\mathsf{k}}\,]) = [\![\mathbf{ET}(M)]\!][\,v^{\mathsf{k}}\,]$.

Proof. First, we show that $[\![\mathbf{ET}(M)]\!][\,v^{\mathsf{k}}\,]$ is well defined and is in $\mathbf{Tree}_{S^{\Xi}_{[\![A^{\mathrm{fin}}]\!]}}$ by coinduction. By Lemma 16 with $\emptyset \vdash M : \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}$, we can find that $\mathbf{ET}(M) \in \mathbf{Tree}_{S^{\Sigma}_{T}}$. We proceed by case analysis on $\mathbf{ET}(M)$.

Case $\mathbf{ET}(M) = \bot$: Obvious because $[\![\mathbf{ET}(M)]\!][\,v^{\mathsf{k}}\,] = [\![\bot]\!][\,v^{\mathsf{k}}\,] = \bot$.

Case $\exists\, V.\ \mathbf{ET}(M) = \mathsf{return}\ V$: Because $[\![\mathbf{ET}(M)]\!][\,v^{\mathsf{k}}\,] = [\![\mathsf{return}\ V]\!][\,v^{\mathsf{k}}\,] = \mathbf{ET}(v^{\mathsf{k}}\, [\![V]\!])$, it suffices to show that

$$\mathbf{ET}(v^{\mathsf{k}}\, [\![V]\!]) \in \mathbf{Tree}_{S^{\Xi}_{[\![A^{\mathrm{fin}}]\!]}} .$$

Because $\mathbf{ET}(M) = \mathsf{return}\ V$, we have $M \longrightarrow^* \mathsf{return}\ V$. By Lemma 5, $\emptyset \vdash \mathsf{return}\ V : \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}$. By its inversion, $\emptyset \vdash V : T$ and $A^{\mathrm{ini}} = A^{\mathrm{fin}}$. By Lemma 11, $\Xi \parallel \emptyset \vdash [\![V]\!] : [\![T]\!]$. Because $\Xi \parallel \emptyset \vdash v^{\mathsf{k}} : [\![T]\!] \to [\![A^{\mathrm{ini}}]\!]$, we have $\Xi \parallel \emptyset \vdash v^{\mathsf{k}}\, [\![V]\!] : [\![A^{\mathrm{ini}}]\!]$ by (T_APP). Thus, by Lemma 18, $\mathbf{ET}(v^{\mathsf{k}}\, [\![V]\!]) \in \mathbf{Tree}_{S^{\Xi}_{[\![A^{\mathrm{ini}}]\!]}}$. Because $A^{\mathrm{ini}} = A^{\mathrm{fin}}$, we have the conclusion.

Case $\exists\, \sigma, c, M_1, \cdots, M_m.\ \mathbf{ET}(M) = \sigma(c, \mathbf{ET}(M_1), \cdots, \mathbf{ET}(M_m))$: By the definition of $\mathbf{ET}(M)$, we have $\sigma = \sigma_i$ for some i such that $E_i = \mathsf{m}$. Because

$$[\![\mathbf{ET}(M)]\!][\,v^{\mathsf{k}}\,] = [\![\sigma_i(c, \mathbf{ET}(M_1), \cdots, \mathbf{ET}(M_m))]\!][\,v^{\mathsf{k}}\,] = \sigma_i(c, [\![\mathbf{ET}(M_1)]\!][\,v^{\mathsf{k}}\,], \cdots, [\![\mathbf{ET}(M_m)]\!][\,v^{\mathsf{k}}\,]) ,$$

it suffices to show that, for any $j \in [1, m]$, $[\![\mathbf{ET}(M_j)]\!][\,v^{\mathsf{k}}\,] \in \mathbf{Tree}_{S^{\Xi}_{[\![A^{\mathrm{fin}}]\!]}}$. By the coinduction hypothesis, it suffices to show that, for any $j \in [1, m]$,

$$\emptyset \vdash M_j : \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}} .$$

Let $j \in [1, m]$. Because $\mathbf{ET}(M) = \sigma_i(c, \mathbf{ET}(M_1), \cdots, \mathbf{ET}(M_m))$ and $\emptyset \vdash M : \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}$, we have $M \longrightarrow^* \sigma_i(c; x.\, M')$ for some x and M' such that $M_j = M'[\underline{j}/x]$. By Lemma 5, $\emptyset \vdash \sigma_i(c; x.\, M') : \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}$. By its inversion, $x : \mathsf{m} \vdash M' : \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow T_i$ and $\underline{A}^{\mathrm{fin}} = T_i$. Because $\emptyset \vdash \underline{j} : \mathsf{m}$ by (HT_ECONST), we have $\emptyset \vdash M'[\underline{j}/x] : \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow T_i$ by Lemma 2. Because $A^{\mathrm{fin}} = T_i$, we have the conclusion.

Next, we show that $\mathbf{ET}(\llbracket M \rrbracket[\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,])$ is well defined and is in $\mathbf{Tree}_{S^{\Xi}_{\llbracket A^{\mathrm{fin}} \rrbracket}}$. We have $\Xi \parallel \emptyset \vdash \overline{v^{\mathsf{h}}} : \Sigma$ because we can derive $\Xi \parallel \emptyset \vdash \lambda x, k.\sigma_i(x; y.\, k\, y) : B_i \to (E_i \to \llbracket T_i \rrbracket) \to \llbracket T_i \rrbracket$ for any $i \in [1, n]$ as follows:

$$\dfrac{\Xi \ni \sigma_i : B_i \rightsquigarrow E_i \qquad \Xi \parallel \Delta \vdash x : B_i \qquad \dfrac{\dfrac{}{\Xi \parallel \Delta' \vdash k : E_i \to \llbracket T_i \rrbracket}\,(\text{T\_Var}) \qquad \dfrac{}{\Xi \parallel \Delta' \vdash y : E_i}\,(\text{T\_Var})}{\Xi \parallel \Delta' \vdash k\, y : \llbracket T_i \rrbracket}\,(\text{T\_App})}{\dfrac{\Xi \parallel \Delta \vdash \sigma_i(x; y.\, k\, y) : \llbracket T_i \rrbracket}{\Xi \parallel \emptyset \vdash \lambda x, k.\sigma_i(x; y.\, k\, y) : B_i \to (E_i \to \llbracket T_i \rrbracket) \to \llbracket T_i \rrbracket}\,(\text{T\_Abs}),(\text{T\_Return})}\,(\text{T\_Op})$$

where $\Delta = x : B_i, k : E_i \to \llbracket T_i \rrbracket$ and $\Delta' = \Delta, y : E_i$. Note that $\llbracket B \rrbracket = B$ and $\llbracket E \rrbracket = E$ for any B and E. Thus, by Lemma 11, $\Xi \parallel \emptyset \vdash \llbracket M \rrbracket[\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] : \llbracket A^{\mathrm{fin}} \rrbracket$. Therefore, $\mathbf{ET}(\llbracket M \rrbracket[\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]) \in \mathbf{Tree}_{S^{\Xi}_{\llbracket A^{\mathrm{fin}} \rrbracket}}$ by Lemma 18.

Finally, we show that $\mathbf{ET}(\llbracket M \rrbracket[\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]) = \llbracket \mathbf{ET}(M) \rrbracket[\,v^{\mathsf{k}}\,]$ by coinduction. We proceed by case analysis on the evaluation of M.

Case $M \longrightarrow^{\omega}$: By Lemmas 5, 15, and 14, $\llbracket M \rrbracket[\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] \longrightarrow^{\omega}$. Therefore, $\llbracket \mathbf{ET}(M) \rrbracket[\,v^{\mathsf{k}}\,] = \mathbf{ET}(\llbracket M \rrbracket[\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]) = \bot$.

Case $\exists\, V.\ M \longrightarrow^{*} \mathsf{return}\ V$: By Lemmas 5, 15 and 14, $\llbracket M \rrbracket[\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] \longrightarrow^{*} \llbracket \mathsf{return}\ V \rrbracket[\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] = v^{\mathsf{k}}\, \llbracket V \rrbracket$. Because $\Xi \parallel \emptyset \vdash \llbracket M \rrbracket[\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] : \llbracket A^{\mathrm{fin}} \rrbracket$ as shown above, we have $\mathbf{ET}(\llbracket M \rrbracket[\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]) = \mathbf{ET}(v^{\mathsf{k}}\, \llbracket V \rrbracket)$ by Lemma 19. Because $\llbracket \mathbf{ET}(M) \rrbracket[\,v^{\mathsf{k}}\,] = \llbracket \mathsf{return}\ V \rrbracket[\,v^{\mathsf{k}}\,] = \mathbf{ET}(v^{\mathsf{k}}\, \llbracket V \rrbracket)$, we have the conclusion.

Case $\exists \sigma, V, z, M'.\ M \longrightarrow^{*} \sigma(V; z.\, M')$: Because $\mathbf{ET}(M) \in \mathbf{Tree}_{S^{\Sigma}_{T}}$ by Lemma 16, we have $V = c$ and $\sigma = \sigma_i$ for some c and i. By Lemmas 5, 15, and 14, $\emptyset \vdash \sigma_i(c; z.\, M') : \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}$ and $\llbracket M \rrbracket[\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] \longrightarrow^{*} \llbracket \sigma_i(c; z.\, M') \rrbracket[\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]$. Thus,

$$\begin{array}{rcl} \llbracket M \rrbracket[\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] & \longrightarrow^{*} & \llbracket \sigma_i(c; z.\, M') \rrbracket[\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] \\ & = & v^{\mathsf{h}}_i\; c\; \lambda z. \llbracket M' \rrbracket[\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] \quad (\text{note that } \sigma_i : B_i \rightsquigarrow E_i \,/\, T_i \Rightarrow T_i \in \Sigma) \\ & = & (\lambda x, k.\sigma_i(x; y.\, k\, y))\; c\; \lambda z. \llbracket M' \rrbracket[\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] \\ & \longrightarrow^{*} & \sigma_i(c; y.\, (\lambda z. \llbracket M' \rrbracket[\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,])\, y)\ . \end{array}$$

Let $E_i = \mathsf{m}$ for some m. Then,

$$\begin{array}{rcl} \mathbf{ET}(\llbracket M \rrbracket[\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]) & = & \mathbf{ET}(\sigma_i(c; y.\, (\lambda z. \llbracket M' \rrbracket[\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,])\, y)) \quad (\text{by Lemma 19 with } \Xi \parallel \emptyset \vdash \llbracket M \rrbracket[\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,] : \llbracket A^{\mathrm{fin}} \rrbracket) \\ & = & \sigma_i(c, \mathbf{ET}((\lambda z. \llbracket M' \rrbracket[\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,])\, \underline{1}), \cdots, \mathbf{ET}((\lambda z. \llbracket M' \rrbracket[\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,])\, \underline{\mathsf{m}})) \\ & = & \sigma_i(c, \mathbf{ET}(\llbracket M' \rrbracket[\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,][\underline{1}/z]), \cdots, \mathbf{ET}(\llbracket M' \rrbracket[\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,][\underline{\mathsf{m}}/z])) \quad (\text{by Lemma 19}) \\ & = & \sigma_i(c, \mathbf{ET}(\llbracket M'[\underline{1}/z] \rrbracket[\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]), \cdots, \mathbf{ET}(\llbracket M'[\underline{\mathsf{m}}/z] \rrbracket[\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,])) \quad (\text{by Lemma 12}) \end{array}$$

(note that, for any $j \in [1, m]$, $(\lambda z. \llbracket M' \rrbracket[\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,])\, \underline{\mathsf{j}}$ is well typed by Lemmas 10 and 7). On the other hand,

$$\begin{array}{rcl} \llbracket \mathbf{ET}(M) \rrbracket[\,v^{\mathsf{k}}\,] & = & \llbracket \sigma_i(c, \mathbf{ET}(M'[\underline{1}/z]), \cdots, \mathbf{ET}(M'[\underline{\mathsf{m}}/z])) \rrbracket[\,v^{\mathsf{k}}\,] \\ & = & \sigma_i(c, \llbracket \mathbf{ET}(M'[\underline{1}/z]) \rrbracket[\,v^{\mathsf{k}}\,], \cdots, \llbracket \mathbf{ET}(M'[\underline{\mathsf{m}}/z]) \rrbracket[\,v^{\mathsf{k}}\,])\ . \end{array}$$

Let $j \in [1, m]$. Now, it suffices to show that

$$\mathbf{ET}(\llbracket M'[\underline{\mathsf{j}}/z] \rrbracket[\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]) = \mathbf{ET}(M'[\underline{\mathsf{j}}/z])\ .$$

By the coinduction principle, it suffices to show that

$$\emptyset \vdash M'[\underline{\mathsf{j}}/z] : \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}\ .$$

It is shown by the inversion of $\emptyset \vdash \sigma_i(c; z.\, M') : \Sigma \triangleright T \,/\, A^{\mathrm{ini}} \Rightarrow A^{\mathrm{fin}}$ and Lemma 2 with $\emptyset \vdash \underline{\mathsf{j}} : \mathsf{m}$.

Otherwise: Contradictory with Lemmas 4 and 5.

□

Theorem 1 (Preservation of Effect Trees)**.** *Let* $\Sigma = \{\sigma_i : B_i \rightsquigarrow E_i \,/\, T_i \Rightarrow T_i\}^{1 \le i \le n}$ *and* T *be a ground type. Assume that* $\emptyset \vdash M : \Sigma \triangleright T \,/\, T \Rightarrow A^{\mathrm{fin}}$ *and* $\sigma_1, \cdots, \sigma_n$ *are ordered. Let* $\overline{v^{\mathsf{h}}} = v^{\mathsf{h}}_1, \cdots, v^{\mathsf{h}}_n$ *such that, for any* $i \in [1, n]$, $v^{\mathsf{h}}_i = \lambda x, k.\sigma_i(x; y.\, k\, y)$ *for some distinct variables* x, k, *and* y. *Also, let* $v^{\mathsf{k}} = \lambda x.\mathsf{return}\, x$. *Then,* $\mathbf{ET}([\![M]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]) = \mathbf{ET}(M)$.

Proof. Let $\Xi = \{\sigma_i : B_i \rightsquigarrow E_i\}^{1 \le i \le n}$. By (T_Var), (T_Return), and (T_Abs), we have $\Xi \parallel \emptyset \vdash v^{\mathsf{k}} : [\![T]\!] \to [\![T]\!]$. Thus, by Lemma 20, $\mathbf{ET}([\![M]\!][\,\overline{v^{\mathsf{h}}} \mid v^{\mathsf{k}}\,]) = [\![\mathbf{ET}(M)]\!][\,v^{\mathsf{k}}\,]$. Then, it suffices to show that

$$[\![\mathbf{ET}(M)]\!][\,v^{\mathsf{k}}\,] = \mathbf{ET}(M)\ .$$

We show it by coinduction. By case analysis on the evaluation of M.

Case $M \longrightarrow^{\omega}$: Obvious because $[\![\mathbf{ET}(M)]\!][\,v^{\mathsf{k}}\,] = \mathbf{ET}(M) = \bot$.

Case $\exists\, V.\ M \longrightarrow^{*} \mathsf{return}\ V$: By the definition, $\mathbf{ET}(M) = \mathsf{return}\ V$ and $[\![\mathbf{ET}(M)]\!][\,v^{\mathsf{k}}\,] = [\![\mathsf{return}\ V]\!][\,v^{\mathsf{k}}\,] = \mathbf{ET}(v^{\mathsf{k}}\,[\![V]\!]) = \mathsf{return}\,[\![V]\!]$. By Lemma 5 with $\emptyset \vdash M : \Sigma \triangleright T \,/\, T \Rightarrow A^{\mathrm{fin}}$, we have $\emptyset \vdash \mathsf{return}\ V : \Sigma \triangleright T \,/\, T \Rightarrow A^{\mathrm{fin}}$. By its inversion, $\emptyset \vdash V : T$. Because T is ground, $V = c$ for some c, or $V = \underline{\mathsf{i}}$ for some i by Lemma 3. In both cases, $[\![V]\!] = V$. Thus, we have the conclusion.

Case $\exists\, \sigma, V, x, M'.\ M \longrightarrow^{*} \sigma(V; x.\, M')$: Because $\mathbf{ET}(M) \in \mathbf{Tree}_{S^{\Sigma}_{T}}$ by Lemma 16, we have $\sigma = \sigma_i$ and $V = c$ for some i and c. Let $E_i = \mathsf{m}$ for some m. By the definition, $\mathbf{ET}(M) = \sigma_i(c, \mathbf{ET}(M'[\underline{1}/x]), \cdots, \mathbf{ET}(M'[\underline{\mathsf{m}}/x]))$ and $[\![\mathbf{ET}(M)]\!][\,v^{\mathsf{k}}\,] = \sigma_i(c, [\![\mathbf{ET}(M'[\underline{1}/x])]\!][\,v^{\mathsf{k}}\,], \cdots, [\![\mathbf{ET}(M'[\underline{\mathsf{m}}/x])]\!][\,v^{\mathsf{k}}\,])$. Let $j \in [1, m]$. It suffices to show that $\mathbf{ET}(M'[\underline{\mathsf{j}}/x]) = [\![\mathbf{ET}(M'[\underline{\mathsf{j}}/x])]\!][\,v^{\mathsf{k}}\,]$. By the coinduction principle, it suffices show that

$$\emptyset \vdash M'[\underline{\mathsf{j}}/x] : \Sigma \triangleright T \,/\, T \Rightarrow A^{\mathrm{fin}}\ .$$

By Lemma 5 with $\emptyset \vdash M : \Sigma \triangleright T \,/\, T \Rightarrow A^{\mathrm{fin}}$ and $M \longrightarrow^{*} \sigma_i(c; x.\, M')$, we have $\emptyset \vdash \sigma_i(c; x.\, M') : \Sigma \triangleright T \,/\, T \Rightarrow A^{\mathrm{fin}}$. By its inversion, $x : \mathsf{m} \vdash M' : \Sigma \triangleright T \,/\, T \Rightarrow A^{\mathrm{fin}}$ (note that $A^{\mathrm{fin}} = T_i$). Because $\emptyset \vdash \underline{\mathsf{j}} : \mathsf{m}$ by (HT_EConst), we have the conclusion by Lemma 2.

Otherwise: Contradictory with Lemmas 5 and 4.

□