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Short Paper

Spherical Blue Noise

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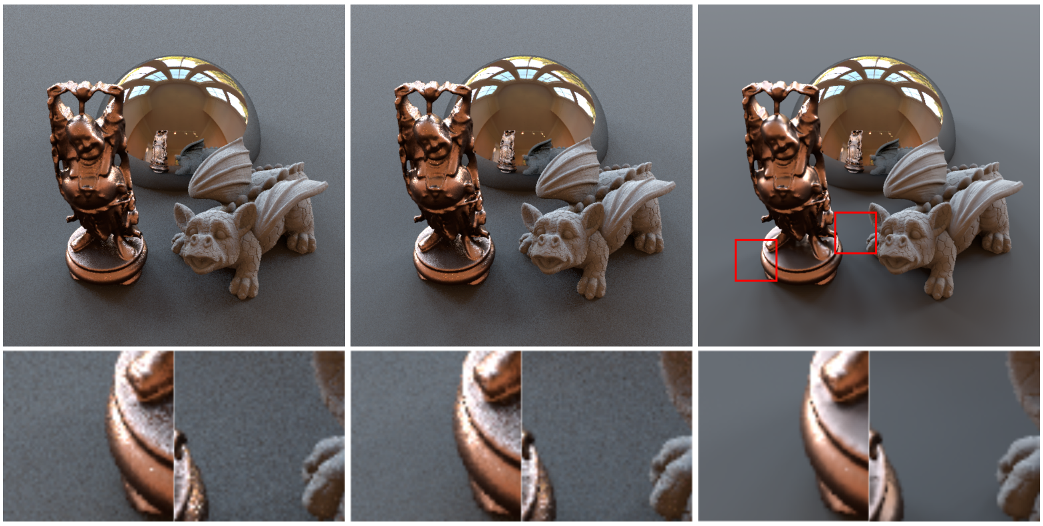

(a) *Spherical Fibonacci, SSIM=0.8717.* **(b)** *Ours (q_s=90), SSIM=0.8973.* **(c)** *Reference (8192 light samples).*

Figure 1: *Comparison of our point set with the spherical Fibonacci point set in 64 light samples rendering.*

Abstract

We present a physically based method which generates unstructured uniform point set directly on the S2-sphere. Spherical uniform point sets are useful for illumination sampling in Quasi Monte Carlo (QMC) rendering but it is challenging to generate high quality uniform point sets directly. Most methods rely on mapping the low discrepancy unit square point sets to the spherical domain. However, these transformed point sets often exhibit sub-optimal uniformity due to the inability of preserving the low discrepancy properties. Our method is designed specifically for direct generation of uniform point sets in the spherical domain. We name our generated result as Spherical Blue Noise point set because it shares similar point distribution characteristics with the 2D blue noise. Our point sets possess high spatial uniformity without a global structure, and we show that they deliver competitive results for illumination integration in QMC rendering, and general numerical integration on the spherical domain.

CCS Concepts
•***Computing methodologies*** → ***Ray tracing;***

1. Introduction

Quasi Monte Carlo rendering methods [Kel13] rely on the uniform point sets for sphere as sampling pattern to estimate the illumination integral. These point sets are often produced by transforming the low discrepancy point sets [S*91, WLH97] designed for the unit square. There exist many mapping techniques [SC97] which minimize the distortion of transforming the unit square point set to the unit sphere but most of them fail to preserve the original low discrepancy properties. Marques et al. [MBR*13] has recently pro-

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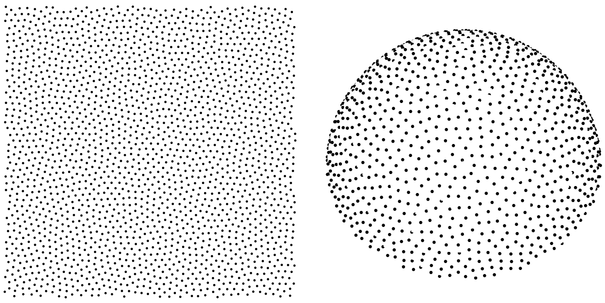
posed to use the spherical Fibonacci point set for illumination integral, and it demonstrates the advantage of a native uniform point set for sphere over the transformed point sets.



(a) *2D Blue Noise.* **(b)** *Spherical Blue Noise.*

Figure 2: *Examples of blue noise point sets.*

Blue noise [Uli88] is a well known sampling pattern for computer graphics applications. This point set possesses good uniformity without a global structure but its use is often limited to 2D sampling. In this paper, we present a direct uniform point set generation method for the sphere using a physically based model, and we name the produced point set as *Spherical Blue Noise* as it shares the essential characteristics of blue noise point set on the unit square. Fig. 2 shows an example of 2D blue noise and our spherical blue noise point set.

2. Spherical Blue Noise

In this section, we describe our physically based model for uniform point set generation on the sphere, and our simulation based implementation in detail.

2.1. Physically based model

We propose to model the members of a uniform point set on the unit sphere as a collection of free-moving electrically charged particles on the spherical surface. When these particles carry identical charges, each of them experiences repulsive force from its neighbors, and moves accordingly. The whole system in motion can be considered as undergoing a self-organizing process to minimize the force experienced by each particle. Once the system reaches the stationary state, each particle should be found maintaining an equidistant neighborhood, i.e. same distance to all its neighbors. A similar physical model has been proposed by Wong and Wong [WW17] whereas they model uniform point sets for the 2D plane.

We revisit quickly the simplified electrostatic model presented by [WW17]. They proposed the net force F_i experienced by each particle i of a N-particle collection on the 2D plane as follows:

$$F_i = q_s^2 \sum_{j \neq i}^{N} \frac{1}{\|r_i - r_j\|^2} \hat{e}_{j,i} \tag{1}$$

where r_i and r_j are the positions of particles i and j respectively; $\hat{e}_{j,i}$ is a unit vector pointing from r_j to r_i which represents the direction of repulsive force, and q_s is the amount of electrical charge carried by each particle.

For a particle on the unit sphere, we notice that its motion can always be regarded as the angular change with respect to an axis of rotation. As shown in Fig. 3, the distance between two particles A and B, can also be measured as the angle θ subtended by their position vectors. Furthermore, particles A and B experience repulsive force from each other, and their motions caused by such force can be mathematically represented by the angular acceleration vectors $\hat{\alpha}_A$ and $\hat{\alpha}_B$ respectively. In short, we can formulate the system of motion of these particles on the unit sphere by using the position vector together with angular velocity and angular acceleration vectors. We propose to transform the interaction model for 2D particles [WW17] in equation 1 to the unit sphere using the angular motion formulation. Our N-particle interaction model on the unit sphere, i.e. the net angular force F_i exerted on particle i can be expressed as follows:

$$F_i = q_s^2 \sum_{j \neq i}^{N} \frac{1}{\|\theta_{i,j}\|^2} \hat{\alpha}_{j,i} \tag{2}$$

where $\theta_{i,j}$ is the angle subtended by the position vectors of particles i and j; and $\hat{\alpha}_{j,i}$ is an angular acceleration unit vector which represents the direction of repulsive force, and q_s is the amount of charge carried by each particle. The charge q_s governs the distribution of the generated point set similarly as in [WW17].

$$F_i = q_s^2 \sum_{j \neq i}^{N} cos(\theta_{i,j}) \hat{\alpha}_{j,i} \tag{3}$$

According to our experiments, the force model is not limited to the inverse square law only. Any isotropic force model with a smooth drop-off characteristic should work reasonably well. We find that a force model based on the cosine of the subtended angle (equation 3) works very well, and its more gentle repulsion helps to stabilize the convergence further.

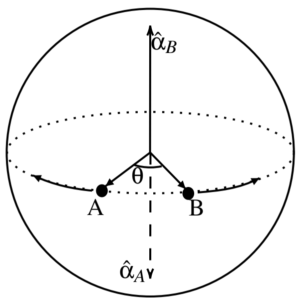


Figure 3: *Motion of particles on the unit sphere formulated as angular motions.*

2.2. Algorithm and Implementation

Based of the system of motion derived in last section, we may track the positions of the particles by integrating the equations of motion. We use a customized Verlet integration algorithm [SABW82] as shown in Algorithm **1**. We apply two modifications to enhance the stability and convergence speed of the algorithm; they are the maximum angular displacement threshold **D** applied in step (1) and a factor **S** which reduces the sustained angular velocity in step (3).

Algorithm 1 Integration algorithm

1. **Position update:**
   $\vec{x}(t+\delta t) = \vec{x}(t) + min(\mathbf{D}, \vec{\omega}(t)\delta t + \frac{1}{2}\vec{\alpha}(t)\delta t^2)$
2. **Angular acceleration update:**
   compute $\vec{\alpha}(t+\delta t)$ using $\vec{x}(t+\delta t)$ via equation (3)
3. **Angular velocity update:**
   $\vec{\omega}(t+\delta t) = min(\frac{\mathbf{D}}{\delta t}, \mathbf{S}\vec{\omega}(t) + \frac{1}{2}(\vec{\alpha}(t) + \vec{\alpha}(t+\delta t))\delta t)$
4. **Repeat**

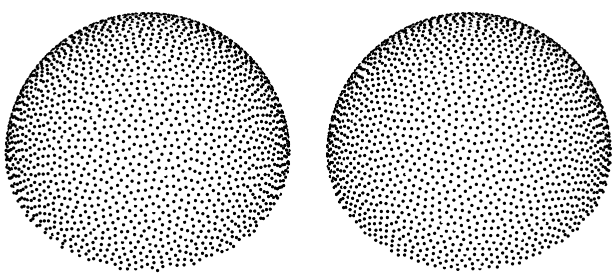
We implement our simulation engine using OpenGL GLSL compute shader. In our simulations, we find a value of $1.6 \times 10^{-3} rad.$ for **D**, and a value of 0.95 for **S** work adequately in all cases. For a 4,096-point set with $q_s = 15$, each iteration takes approximately 1.7 milliseconds on a commodity GPU, and it takes 1,000 iterations to converge. However, the number of iterations depends on the initial positions of the particles, and it can be shortened considerably by using a roughly uniform point set to initialize particles' positions.

3. Results and Evaluations

In this section, we demonstrate the point distribution behavior of our blue noise point sets, and compare our results with other commonly used uniform point sets for the unit sphere. We also assess the quality of our point set for illumination integration in QMC rendering and general spherical quadrature.

3.1. Spherical blue noise samples distribution

In our model, the electrical charge q_s has a direct impact on the repulsive force exerted on the particles, and this quantity controls directly the point distribution pattern of the resultant point set. Fig. 4 shows two spherical blue noise point sets generated using different magnitudes of the electrical charge q_s.



(a) *Charge,* $q_s = 30.0$. **(b)** *Charge,* $q_s = 80.0$.

Figure 4: *Impact of the electrical charge q_s to the generated spherical 2048-point blue noise point sets.*

The weak electrical charge of particles (Fig. 4a) results a force field which dissipates more quickly, and the particles are only sensitive to its smaller neighborhood. In contrast, a stronger charge (Fig. 4b) results a force field which affects a bigger neighborhood, and the particles become more structured locally with a tendency of reaching hexagonal packing [SK97].

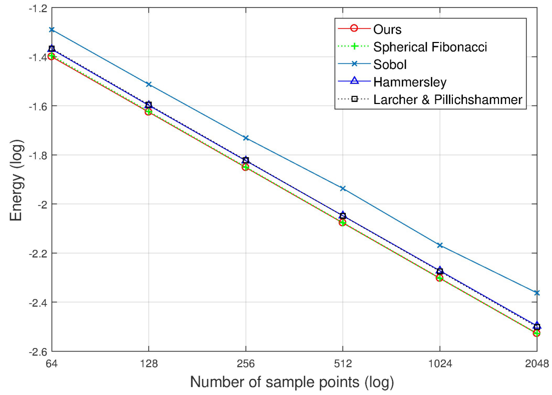


Figure 5: *Distance based energy of various point sets.*

3.2. Point set quality for spherical integration

We apply the energy metric proposed by [BD12,BSSW14] to assess the quality of our point set for spherical numerical integration. The distance based energy metric of a point set is given by

$$E_N(P_s) = \left(\frac{4}{3} - \frac{1}{N^2} \sum_{j=1}^{N} \sum_{i=1}^{N} \|r_i - r_j\| \right)^{\frac{1}{2}} \quad (4)$$

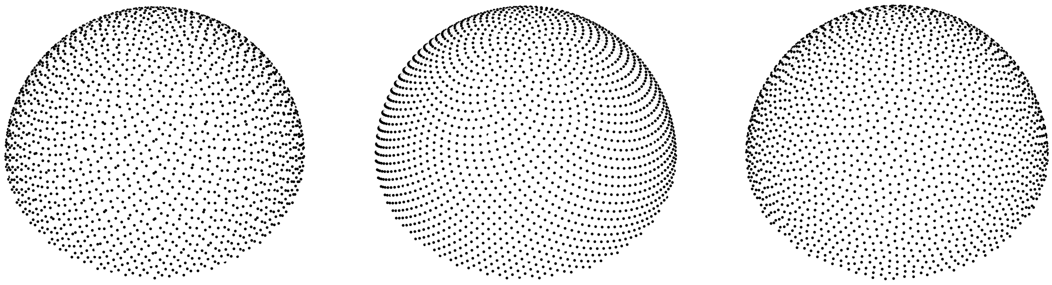
and a lower energy value implies better performance for spherical quadrature. Fig. 5 shows the distance based energy plot of different sample size of our point sets and other commonly used point sets for spherical quadrature. The Sobol [Sob67], Hammersley [WLH97] and [LP01] point sets are transformed point sets, and since their uniformity are all sub-optimal (see Fig. 6), they all have higher energy than ours and the spherical Fibonacci [MBR*13] point sets. Our point sets deliver similar integration quality as spherical Fibonacci point sets but we perform consistently better for low sample counts, and Table 1 compares the energy of our best point sets with the spherical Fibonacci point sets in detail.

Table 1: *Distance based energy ($\times 10^{-3}$, lower is better)*

Sample size	Spherical Fibonacci	Ours	q_s
64	4.016	**3.976**	120
128	2.379	**2.366**	115
256	1.411	**1.407**	105
512	0.838	**0.837**	115
1024	0.497	0.497	180
2048	0.297	**0.296**	170

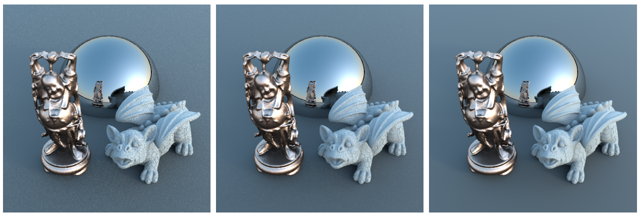
3.3. Point set quality for illumination integration

As the distance based energy indicates that our point set performs better in lower sample count, we validate this result with an application of our point sets in QMC rendering. Fig. 1 and 7 show the rendering results, and the perceptual metric SSIM indicates that our rendering results are perceptually more similar to the reference renderings.



(a) *Hammersley (mapped).* **(b)** *Spherical Fibonacci.* **(c)** *Ours ($q_s = 70$).*

Figure 6: *Comparison of 2,048-point sets generated by different methods.*



(a) *Spherical Fibonacci, SSIM=0.9216.* **(b)** *Ours (q_s=130), SSIM=0.9580.* **(c)** *Reference, (8192 light samples).*

Figure 7: *Illumination integration quality with 32 light samples.*

4. Discussions

Fig. 6 shows three different uniform point sets for the unit sphere. Hammersley (Fig. 6a) is commonly used in QMC rendering but its uniformity is obviously imperfect. Spherical Fibonacci point set (Fig. 6b) recently introduced by [MBR*13] has good uniformity and low distance-based energy. However, spherical Fibonacci point set possesses an axis of formation and it is seemingly more structured especially in the areas away from the axis. There may be some potential issues on certain sampling problems such as temporal aliasing but we have not pursued further.

Our spherical blue noise (Fig. 6c) has good uniformity, and the absence of a global structure makes our point set potentially more resilient to certain aliasing problems as blue noise sampling is known to be capable of transforming high frequency aliases into noise. One obvious limitation of our method is its computationally expensive simulation step when compared with other methods.

5. Conclusions

We have presented a novel physically based method for direct generation of uniform point sets for the unit sphere. Our method has a user parameter (electrical charge) for fine-tuning the distribution. Our spherical blue noise point sets deliver on par or better integration results than the state-of-the-art method especially for the low sampling rate scenarios.

6. Acknowledgement

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