Morphisms Between Pattern Structures and Their Impact on Concept Lattices

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Abstract. We provide a general framework for pattern structures by investigating adjunctions between posets and their morphisms. Our special interest is the impact of pattern morphisms on the induced concept lattices. In particular we are interested in conditions which are sufficient for the induced residuated maps to be injective, surjective or bijective. One application is that every representation context of a pattern structure has a formal concept lattice that is induced by a certain pattern morphism.

1 Introduction

Pattern structures within the framework of formal concept analysis have been introduced in [3]. Since then they have turned out to be a useful tool for analysing various real-world applications (cf. [3–7]). Our paper extends the concept of representation contexts and interprets them via morphisms, closely related to o-projections as recently introduced and investigated in [2]. In [8], we disussed the meaning of projections of pattern structures, realizing the importance of residual projections. As a matter of fact, our generalization of representation contexts of pattern structures gives rise to residual projections.

2 Preliminaries

The fundamental order theoretic concepts of our paper are nicely presented in the book on *Residuation Theory* by T.S. Blythe and M.F. Janowitz (cf. [1]).

Definition 1 (Adjunction). *Let $\mathbb{P} = (P,\leqslant)$ and $\mathbb{L} = (L,\leqslant)$ be posets; furthermore let $f : P \to L$ and $g : L \to P$ be maps.*

(1) The pair (f,g) is an **adjunction** *w.r.t. $(\mathbb{P},\mathbb{L})$ if $fx \leqslant y$ is equivalent to $x \leqslant gy$ for all $x \in P$ and $y \in L$. In this case, we will refer to $(\mathbb{P},\mathbb{L},f,g)$ as a* **poset adjunction**.
(2) f is **residuated** *from $\mathbb{P}$ to $\mathbb{L}$ if the preimage of a principal ideal in $\mathbb{L}$ under f is always a principal ideal in $\mathbb{P}$, that is, for every $y \in L$ there exists $x \in P$ s.t.*

$$f^{-1}\{t \in L \mid t \leqslant y\} = \{s \in P \mid s \leqslant x\}.$$

(3) g is **residual** *from $\mathbb{L}$ to $\mathbb{P}$ if the preimage of a principal filter in $\mathbb{P}$ under g is always a principal filter in $\mathbb{L}$, that is, for every $x \in P$ there exists $y \in L$ s.t.*

$$g^{-1}\{s \in P \mid x \leqslant s\} = \{t \in L \mid y \leqslant t\}.$$

(4) The dual of $\mathbb{L}$ is given by $\mathbb{L}^{\mathrm{op}} = (L,\geqslant)$ with $\geqslant := \{(x,t) \in L \times L \mid t \leqslant x\}$. The pair (f,g) is a **Galois connection** *w.r.t. $(\mathbb{P},\mathbb{L})$ if (f,g) is an* **adjunction** *w.r.t. $(\mathbb{P},\mathbb{L}^{\mathrm{op}})$.*

The following well-known facts are straightforward (cf. [1]).

Proposition 1. *Let $\mathbb{P} = (P,\leqslant)$ and $\mathbb{L} = (L,\leqslant)$ be posets.*

(1) A map $f : P \to L$ is residuated from $\mathbb{P}$ to $\mathbb{L}$ iff there exists a map $g : L \to P$ s.t. (f,g) is an adjunction w.r.t. $(\mathbb{P},\mathbb{L})$.
(2) A map $g : L \to P$ is residual from $\mathbb{L}$ to $\mathbb{P}$ iff there exists a map $f : P \to L$ s.t. (f,g) is an adjunction w.r.t. $(\mathbb{P},\mathbb{L})$.
(3) If (f,g) and (h,k) are adjunctions w.r.t. $(\mathbb{P},\mathbb{L})$ with $f = h$ or $g = k$ then $f = h$ and $g = k$.
(4) If f is a residuated map from $\mathbb{P}$ to $\mathbb{L}$, then there exists a unique residual map f^+ from $\mathbb{L}$ to $\mathbb{P}$ s.t. (f,f^+) is an adjunction w.r.t. $(\mathbb{P}, \mathbb{L})$. In this case, f^+ is called the **residual map** *of f.*
(5) If g is a residual map from $\mathbb{L}$ to $\mathbb{P}$, then there exists a unique residuated map g^- from $\mathbb{P}$ to $\mathbb{L}$ s.t. (g^-,g) is an adjunction w.r.t. $(\mathbb{P}, \mathbb{L})$. In this case, g^- is called the **residuated map** *of g.*
(6) A residuated map f from $\mathbb{P}$ to $\mathbb{L}$ is surjective iff $f \circ f^+ = id_L$ iff f^+ is injective.
(7) A residuated map f from $\mathbb{P}$ to $\mathbb{L}$ is injective iff $f \circ f^+ = id_L$ iff f^+ is surjective.

Definition 2. *Let $\mathbb{P} = (P,\leqslant)$ be a poset and $T \subseteq P$. Then*

(1) The **restriction** *of $\mathbb{P}$ onto T is given by $\mathbb{P}|T := (T, \leqslant \cap (T \times T))$, which clearly is a poset too.*
(2) The **canonical embedding** *of $\mathbb{P}|T$ into $\mathbb{P}$ is given by the map $T \to P, t \mapsto t$.*
(3) T is a **kernel system** *in $\mathbb{P}$ if the canonical embedding τ of $\mathbb{P}|T$ into $\mathbb{P}$ is residuated. In this case, the residual map φ of τ will also be called the* **residual map** *of T in $\mathbb{P}$. The composition $\kappa := \tau \circ \varphi$ is referred to as the* **kernel operator** *associated with T in $\mathbb{P}$.*

(4) *Dually, T is a* **closure system** *in $\mathbb{P}$ if the cannonical embedding τ of $\mathbb{P}|T$ into $\mathbb{P}$ is residual. In this case, the residuated map ψ of τ will also be called the* **residuated map** *of T in $\mathbb{P}$. The composition $\gamma := \tau \circ \psi$ is referred to as the* **closure operator** *associated with T in $\mathbb{P}$.*

(5) *A map $\kappa : P \to P$ is a* **kernel operator** *on $\mathbb{P}$ if $s \leqslant x$ is equivalent to $s \leqslant \kappa x$ for all $s \in \kappa P$ and $x \in P$.*
Remark: In this case, κP forms a kernel system in $\mathbb{P}$, the kernel operator of which is κ.

(6) *Dually, a map $\gamma : P \to P$ is a* **closure operator** *on $\mathbb{P}$ if $x \leqslant t$ is equivalent to $\gamma x \leqslant t$ for all $x \in P$ and $t \in \gamma P$.*
Remark: In this case, φP forms a closure system in $\mathbb{P}$, the closure operator of which is γ.

The following known facts will be needed for the sequel (cf. [1]) .

Proposition 2. *Let $\mathbb{P} = (P, \leqslant)$ and $\mathbb{L} = (L, \leqslant)$ be posets.*

(1) *If f is a residuated map from $\mathbb{P}$ to $\mathbb{L}$ then f preserves all existing suprema in $\mathbb{P}$, that is, if $s \in P$ is the supremum (least upper bound) of $X \subseteq P$ in $\mathbb{P}$ then fs is the supremum of fX in $\mathbb{L}$.*
In case $\mathbb{P}$ and $\mathbb{L}$ are complete lattices, the reverse holds too: If a map f from $\mathbb{P}$ to $\mathbb{L}$ preserves all suprema, that is,

$$f(\sup_{\mathbb{P}} X) = \sup_{\mathbb{L}} fX \ for\ all\ X \subseteq P,$$

then f is residuated.

(2) *If g is a residual map from $\mathbb{L}$ to $\mathbb{P}$, then g preserves all existing infima in $\mathbb{L}$, that is, if $t \in L$ is the infimum (greatest lower bound) of $Y \subseteq L$ in $\mathbb{L}$ then gt is the infimum of gY in $\mathbb{P}$.*
In case $\mathbb{P}$ and $\mathbb{L}$ are complete lattices, the reverse holds too: If a map g from $\mathbb{L}$ to $\mathbb{P}$ preserves all infima, that is,

$$f(\inf_{\mathbb{P}} Y) = \inf_{\mathbb{L}} gY \ for\ all\ Y \subseteq L,$$

then g is residual.

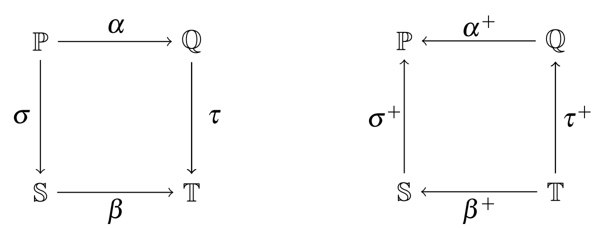
(3) *For an adjunction (f, g) w.r.t. $(\mathbb{P}, \mathbb{L})$ the following hold:*
  - *(a1) f is an isotone map from $\mathbb{P}$ to $\mathbb{L}$.*
  - *(a2) $f \circ g \circ f = f$*
  - *(a3) fP is a kernel system in $\mathbb{L}$ with $f \circ g$ as associated kernel operator on $\mathbb{L}$. In particular, $L \to P, y \mapsto fgy$ is a residual map from $\mathbb{L}$ to $\mathbb{L}|fP$.*
  - *(b1) g is an isotone map from $\mathbb{L}$ to $\mathbb{P}$.*
  - *(b2) $g \circ f \circ g = g$*
  - *(b3) gL is a closure system in $\mathbb{P}$ with $g \circ f$ as associated closure operator on $\mathbb{P}$. In particular, $P \to gL, x \mapsto gfx$ is a residuated map from $\mathbb{P}$ to $\mathbb{P}|gL$.*

3 Adjunctions and Their Concept Posets

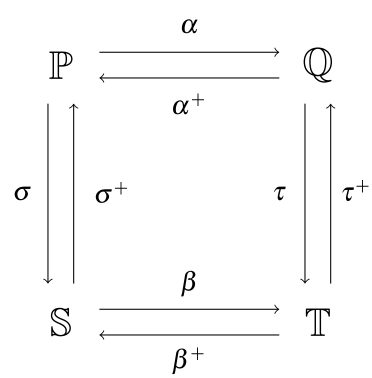
Definition 3. *Let $\mathcal{P} := (\mathbb{P}, \mathbb{S}, \sigma, \sigma^+)$ and $\mathcal{Q} := (\mathbb{Q}, \mathbb{T}, \tau, \tau^+)$ be poset adjunctions. Then a pair (α, β) forms a morphism from $\mathcal{P}$ to $\mathcal{Q}$ if $(\mathbb{P}, \mathbb{Q}, \alpha, \alpha^+)$ and $(\mathbb{S}, \mathbb{T}, \beta, \beta^+)$ are poset adjunctions satisfying*

$$\tau \circ \alpha = \beta \circ \sigma$$

Remark: This implies $\alpha^+ \circ \tau^+ = \sigma^+ \circ \beta^+$, that is, the following diagrams are commutative:



Next we illustrate the involved poset adjunctions:



Definition 4 (Concept Poset)**.** *For a poset adjunction $\mathcal{P} = (\mathbb{P}, \mathbb{S}, \sigma, \sigma^+)$ let*

$$\mathrm{B}\mathcal{P} := \{(p,s) \in P \times S \mid \sigma p = s \wedge \sigma^+ s = p\}$$

denote the set of (**formal**) **concepts** *in $\mathcal{P}$. Then the* **concept poset** *of $\mathcal{P}$ is given by*

$$\mathbb{B}\mathcal{P} := (\mathbb{P} \times \mathbb{S}) \mid \mathrm{B}\mathcal{P},$$

that is, $(p_0, s_0) \leqslant (p_1, s_1)$ holds iff $p_0 \leqslant p_1$ iff $s_0 \leqslant s_1$, for all $(p_0, s_0), (p_1, s_1) \in B\mathcal{P}$. If (p,s) is a formal concept in $\mathcal{P}$ then p is referred to as **extent** *in $\mathcal{P}$ and s as* **intent** *in $\mathcal{P}$.*

From [9] we point out Theorem 1:

Theorem 1. *Let (α, β) be a morphism from a poset adjunction $\mathcal{P} = (\mathbb{P}, \mathbb{S}, \sigma, \sigma^+)$ to a poset adjunction $\mathcal{Q} = (\mathbb{Q}, \mathbb{T}, \tau, \tau^+)$. Then*

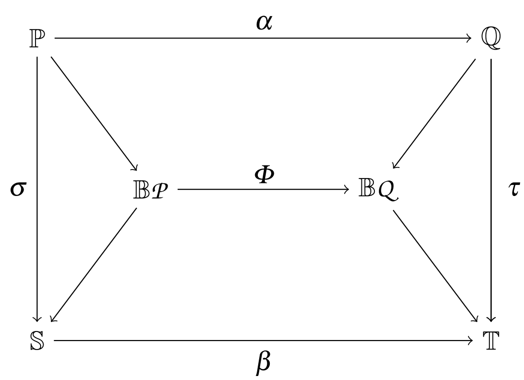
$$(\mathbb{B}\mathcal{P}, \mathbb{B}\mathcal{Q}, \Phi, \Phi^+)$$

is a poset adjunction for

$$\Phi : \mathrm{B}\mathcal{P} \to \mathrm{B}\mathcal{Q}, (p,s) \mapsto (\tau^+\beta s, \beta s)$$

and

$$\Phi^+ : \mathrm{B}\mathcal{Q} \to \mathrm{B}\mathcal{P}, (q,t) \mapsto (\alpha^+ q, \sigma\alpha^+ q).$$



Theorem 2. *Under the conditions of the previous theorem the following hold:*

(1) If α is surjective then Φ is surjective too.
(2) If β is injective then Φ is injective too.
(3) If α is surjective and β is injective then Φ is an isomorphism from $\mathrm{B}\mathcal{P}$ *to* $\mathrm{B}\mathcal{Q}$*.*

Proof. (1) Assume that α is surjective, that is, $\alpha \circ \alpha^+ = id_Q$. Then for all $(p,s) \in \mathrm{B}\mathcal{P}$, the second component of $(\Phi \circ \Phi^+)(p,s)$ is $\beta\sigma\alpha^+ q = \tau\alpha\alpha^+ q = \tau q = s$. This yields $\Phi \circ \Phi^+ = id_{\mathrm{B}\mathcal{Q}}$, that is, Φ is surjective.
(2) The first component of $(\Phi^+ \circ \Phi)(p,s)$ is $\alpha^+\tau{+}\beta s = \tau{+}\beta^+\beta s = \tau{+}s = p$. Therefore, $\Phi^+ \circ \Phi = id_{\mathrm{B}\mathcal{P}}$, which yields Φ being injective.
(3) If α is surjective and β is injective, than Φ and Φ^+ are naturally inverse by (1) and (2), that is, Φ is an isomorphism from $\mathrm{B}\mathcal{P}$ to $\mathrm{B}\mathcal{Q}$.

□

4 The Impact of Pattern Morphism on Concept Lattices

Definition 5. *A triple $\mathcal{G} = (G, \mathbb{D}, \delta)$ is a* **pattern setup** *if G is a set, $\mathbb{D} = (D, \sqsubseteq)$ is a poset, and $\delta : G \to D$ is a map. In case every subset of $\delta G := \{\delta g \mid g \in G\}$ has an infimum in $\mathbb{D}$, we will refer to $\mathcal{G}$ as* **pattern structure***. Then the set*

$$\mathrm{C}_{\mathcal{G}} := \{\inf_{\mathbb{D}} \delta X \mid X \subseteq G\}$$

forms a closure system in $\mathbb{D}$ and furthermore $\mathbb{C}_{\mathcal{G}} := \mathbb{D}|C_{\mathcal{G}}$ forms a complete lattice.

If $\mathcal{G} = (G, \mathbb{D}, \delta)$ and $\mathcal{H} = (H, \mathbb{E}, \varepsilon)$ each is a pattern setup, then a pair (f, φ) forms a **pattern morphism** *from $\mathcal{G}$ to $\mathcal{H}$ if $f : G \to H$ is a map and φ is a residual map from $\mathbb{D}$ to $\mathbb{E}$ satisfying $\varphi \circ \delta = \varepsilon \circ f$, that is, the following diagram is commutative:*

$$
\begin{array}{ccc}
G & \xrightarrow{\;f\;} & H \\
\delta \big\downarrow & & \big\downarrow \varepsilon \\
\mathbb{D} & \xrightarrow[\;\varphi\;]{} & \mathbb{E}
\end{array}
$$

In the sequel we show how our previous considerations apply to pattern structures.

Theorem 3. *Let (f,φ) be a pattern morphism from a pattern structure $\mathcal{G} = (G,\mathbb{D},\delta)$ to a pattern structure $\mathcal{H} = (H,\mathbb{E},\varepsilon)$.*
To apply the previous theorem we give the following construction:
f gives rise to an adjunction (α,α^+) between the power set lattices $\mathbf{2}^G := (2^G,\subseteq)$ and $\mathbf{2}^H := (2^H,\subseteq)$ via

$$\alpha : 2^G \to 2^H, X \mapsto fX$$

and

$$\alpha^+ : 2^H \to 2^G, Y \mapsto f^{-1}Y.$$

Further let φ^- denote the residuated map of φ w.r.t. $(\mathbb{E},\mathbb{D})$, that is, $(\mathbb{E},\mathbb{D},\varphi^-,\varphi)$ is a poset adjunction. Then, obviously, $(\mathbb{D}^{\mathrm{op}},\mathbb{E}^{\mathrm{op}},\varphi,\varphi^-)$ is a poset adjunction too.

For pattern structures the following operators are essential:

$$
\begin{aligned}
{}^{\diamond} &: 2^G \to D, X \mapsto \inf_{\mathbb{D}} \delta X \\
{}^{\blacklozenge} &: D \to 2^G, d \mapsto \{g \in G \mid d \sqsubseteq \delta g\} \\
{}^{\square} &: 2^H \to E, Z \mapsto \inf_{\mathbb{E}} \varepsilon Z \\
{}^{\blacksquare} &: E \to 2^H, e \mapsto \{h \in H \mid e \sqsubseteq \varepsilon h\}
\end{aligned}
$$

It now follows that (α,φ) forms a morphism from the poset adjunction

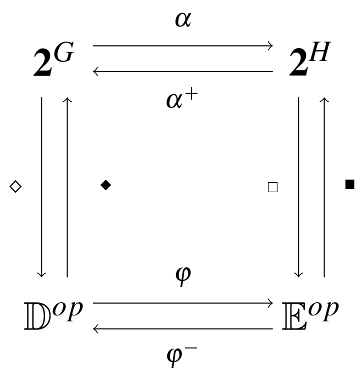
$$\mathcal{P} = (\mathbf{2}^G, \mathbb{D}^{\mathrm{op}}, {}^{\diamond}, {}^{\blacklozenge})$$

to the poset adjunction

$$\mathcal{Q} = (\mathbf{2}^H, \mathbb{E}^{\mathrm{op}}, {}^{\square}, {}^{\blacksquare}).$$

In particular, $(fX)^{\square} = \varphi(X^{\diamond})$ holds for all $X \subseteq G$.

We receive the following diagram of adjunctions:

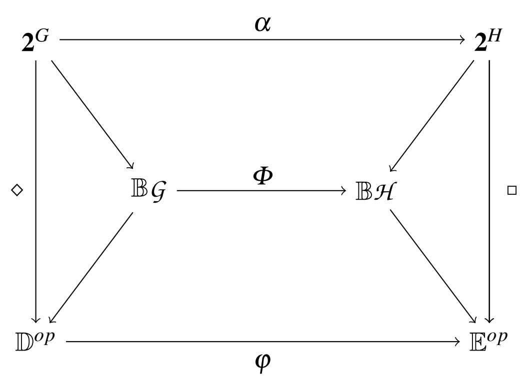


For the following we recall that the concept lattice of $\mathcal{G}$ is given by $\mathbb{B}\mathcal{G} := \mathbb{B}\mathcal{P}$ and the concept lattice of $\mathcal{H}$ is $\mathbb{B}\mathcal{H} := \mathbb{B}\mathcal{Q}$.
Then Theorem 1 yields that the quadruple $(\mathbb{B}\mathcal{G}, \mathbb{B}\mathcal{H}, \Phi, \Phi^+)$ is an adjunction for

$$\Phi : \mathrm{B}\mathcal{G} \to \mathrm{B}\mathcal{H}, (X, d) \mapsto ((\varphi d)^{\blacksquare}, \varphi d)$$

and

$$\Phi^+ : \mathrm{B}\mathcal{H} \to \mathrm{B}\mathcal{G}, (Z, e) \mapsto (f^{-1}Z, (f^{-1}Z)^{\diamond}).$$



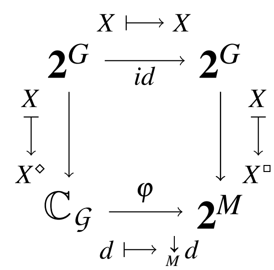
By Theorem 2 the following hold:

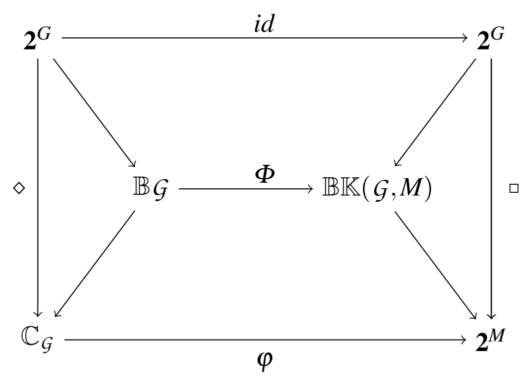
(1) If f is surjective then Φ is surjective too.
(2) If φ is injective then Φ is injective too.
(3) If f is surjective and φ is injective then Φ is an isomorphism from $\mathbb{B}\mathcal{G}$ to $\mathbb{B}\mathcal{H}$.

Theorem 4. *Let $\mathcal{G} = (G, \mathbb{D}, \delta)$ and $\mathcal{H} = (H, \mathbb{E}, \varepsilon)$ be pattern structure. And let $\mathcal{G}^\bullet = (G, \mathbb{C}_{\mathcal{G}}, \delta^\bullet)$ be the pattern structure induced by $\mathcal{G}$ via $\delta^\bullet : G \to C_{\mathcal{G}}$, $g \mapsto \delta g$. It follows $\mathbb{B}\mathcal{G}^\bullet = \mathbb{B}\mathcal{G}$. Further let (f, φ) be a pattern morphism from $\mathcal{G}^\bullet$ to $\mathcal{H}$. Then with the notation introduced in the previous theorem, the map Φ from $\mathbb{B}\mathcal{G}$ to $\mathbb{B}\mathcal{H}$ is residuated. If f is surjective then so is Φ, if φ is injective then so is Φ. If f is surjective and φ is injective then Φ is an isomorphism from $\mathbb{B}\mathcal{G}$ to $\mathbb{B}\mathcal{H}$.*

Definition 6. *The* ***representation context*** *of a pattern structure* $\mathcal{G} = (G, \mathbb{D}, \delta)$ *w.r.t. a subset* M *of* D *is given by* $\mathbb{K}(\mathcal{G}, M) := (G, M, I)$ *with* $I := \{(g,m) \in G \times M \mid m \sqsubseteq \delta g\}$.

Theorem 5. *Let* $\mathcal{G} = (G, \mathbb{D}, \delta)$ *be a pattern structure and let* M *be a subset of* D.
The pattern structure associated with the representation context $\mathbb{K}(\mathcal{G}, M)$ *is given by* $\mathcal{H} := (G, \mathbf{2}^M, \varepsilon)$ *with* $\varepsilon : G \to 2^M, g \mapsto \downarrow_M \delta g$ *where* $\downarrow_M d := \{m \in M \mid m \sqsubseteq d\}$ *for all* $d \in D$.
In particular, the concept lattice of $\mathbb{K}(\mathcal{G}, M)$ *is given by* $\mathbb{B}\mathbb{K}(\mathcal{G}, M) = \mathbb{B}\mathcal{H}$.
Using the notation from the previous theorem, (id_G, φ) *is a pattern morphism from* $\mathcal{G}^\bullet$ *to* $\mathcal{H}$ *for* $\varphi : C_\mathcal{G} \to 2^M, x \mapsto \downarrow_M x$. *Furthermore, the map* Φ *from* $\mathbb{B}\mathcal{G}$ *to* $\mathbb{B}\mathcal{H} = \mathbb{B}\mathbb{K}(\mathcal{G}, M)$ *is a residuated surjection. In case* M *is join-dense w.r.t.* $\mathbb{C}_\mathcal{G}$ *(that is,* φ *is injective),* Φ *is an isomorphism from* $\mathbb{B}\mathcal{G}$ *to* $\mathbb{B}\mathbb{K}(\mathcal{G}, M)$.





Remark: Based on the paradigm of concept morphisms, the previous theorem extends and sheds new light on theorem 1 of [3]. We generalize the definition of representation context introduced in [3] by allowing an abitrary subset M of patterns of the underlying pattern structure $\mathcal{G}$ as attribute set of the representation context $\mathbb{K}(\mathcal{G}, M)$. It then turns out that $\mathbb{K}(\mathcal{G}, M)$ has a formal concept lattice which is induced by a morphism on $\mathcal{G}^\bullet$. More explicitly, there is a morphism on $\mathcal{G}^\bullet$ to the pattern structure of $\mathbb{K}(\mathcal{G}, M)$ which induces a residuated surjection from the concept lattice of $\mathcal{G}$ to the concept lattice of $\mathbb{K}(\mathcal{G}, M)$. In case M is join-dense w.r.t. $\mathcal{G}$, the morphism between the concept lattices is an isomorphism (see also Theorem 1 of [3]). Our extension of the concept of representation context gives rise to various constructions of o-projections (as introduced in [2]) on $\mathcal{G}^\bullet$.

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