From a Possibility Theory View of Formal Concept Analysis to the Possibilistic Handling of Incomplete and Uncertain Contexts

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Abstract. The formal similarity between possibility theory and formal concept analysis, made ten years ago, has suggested the introduction in the latter setting of the counterpart of possibilistic operators, which were ignored before. These new operators can be related to the basic operator of formal concept analysis by a triple use of negations on the contexts, on the set-valued arguments and on the obtained results, and lead to consider new compositions worth of interest. They enable us to complete the Guigues-Duquenne basis with rules having disjunctive conclusions. Besides, the approach can be naturally generalized to incomplete contexts and then to uncertain context where uncertainty is graded.

1 Introduction

Formal Concept Analysis (FCA) considers the classical Galois derivation operator (i.e. the sufficiency operator) for extracting formal concepts organized within a hierarchy (i.e. partial ordering) called the concept lattice. The concept lattice has proved highly useful for knowledge discovery. The knowledge is expressed as attribute implications, that are formulas in the form $\{a_1, ..., a_n\} \rightarrow \{b_1, ..., b_m\}$ where $a_1, ..., a_n, b_1, ...$ and b_m are attributes. It is considered that the underlying semantics is a conjunctive one. Indeed, by $\{a_1, ..., a_n\} \rightarrow \{b_1, ..., b_m\}$, the interpretation "$a_1$" and ... and "$a_n$" $\rightarrow$ "b_1" and ... and "b_m" is implicitly agreed.

Recently, Dubois and Prade [6] [9] have given a possibility-theoretic reading of formal concept analysis. Beyond the sufficiency operator currently used in FCA, the possibilistic interpretation proposed by these authors allows to consider three other (powerset) operators namely possibility, necessity and dual sufficiency [5] [3]. In this spirit, the aim of this paper is to enlarge the knowledge representation capability of FCA to so-called "disjunctive attribute implications" instead of the conjunctive attribute implications considered by current approaches [1] (introduced in [11]). It will be shown that the proposed approach considers "open-closed" pairs obtained by means of the asymmetric composition $(\mathrm{N} \circ \Pi)$ of necessity and possibility operators, and then we propose a method for inducing disjunctive attribute implications.

The remainder of the paper is organized as follows. Section 2 gives a background on FCA. The possibility-theoretic view of FCA is discussed in section 3, whereas the next section presents our contribution which highlights the interest of using possibility theory operators in order to induce disjunctive attribute implications from formal contexts. Section 5 presents the same analysis for incomplete formal contexts and finally, Section 6 deals with necessity degrees in uncertain formal contexts.

2 Formal concept analysis: basic notions

Formal concept analysis [1] is a lattice-based setting for data analysis and knowledge representation. It relies essentially on a binary relation between a set of objects and a set of attributes. This relation is called a formal context. More formally, a formal context is a triple $\mathcal{K} = (\mathcal{O}, \mathcal{P}, \mathcal{R})$ where $\mathcal{O}$ is a set of objects, $\mathcal{P}$ a set of attributes and $\mathcal{R}$ a binary relation s.t. $\mathcal{R} \subseteq \mathcal{O} \times \mathcal{P}$. $x\mathcal{R}a$ means that the object x satisfies the attribute a.

Example 1. We consider an example of formal context $\mathcal{K}_\mathcal{S} = (\mathcal{O}, \mathcal{P}, \mathcal{R})$ given in Table 1 where $\mathcal{O} = \{John, Maria, Peter, Clara\}$ and $\mathcal{P} = \{Man, Woman, Father, Mother, Parent\}$. The cross mark $\times$ indicates that the related object satisfies the corresponding attribute. Whereas the empty mark indicates the contrary.

The paradigm of formal concept analysis [14] is classically based of an adjoint pair of operators $(.)^\Delta : 2^\mathcal{O} \to 2^\mathcal{P}$ and $(.)^\Delta : 2^\mathcal{P} \to 2^\mathcal{O}$ (called Galois derivation operator in the literature) defined for two sets $X \in 2^\mathcal{O}$ and $A \in 2^\mathcal{P}$ as follows :

$$A^\Delta = \{x \in \mathcal{O} \mid \forall a \in \mathcal{P}\ (a \in A \Rightarrow x\mathcal{R}a)\}$$
$$X^\Delta = \{a \in \mathcal{P} \mid \forall x \in \mathcal{O}\ (x \in X \Rightarrow x\mathcal{R}a)\}$$

That is, A^Δ corresponds to the set of objects that satisfy all attributes in A. Similarly, X^Δ corresponds to the set of of attributes that are satisfied by all objects in X.

A formal concept of $\mathcal{K}$ is a pair of closed sets (X, A) with $X \subseteq \mathcal{O}, A \subseteq \mathcal{P}$ such that $X^\Delta = A$ and $A^\Delta = X$. X is called the extent and A the intent of the formal concept (X, A). For instance, ({Clara}, {Woman, Parent, Mother}) is a formal concept of $K_\mathcal{S}$. The set of all formal concepts (denoted by $\mathcal{B}(\mathcal{O}, \mathcal{P}, \mathcal{R})$) equipped with a partial order $\preceq$ defined as: $(X_1, A_1) \preceq (X_2, A_2)$ if $X_1 \subseteq X_2$ (or equivalently, $A_2 \subseteq A_1$) forms a complete lattice (denoted by $\mathfrak{L}(\mathcal{O}, \mathcal{P}, \mathcal{R})$).

Formal concepts lattices can be characterized in terms of attribute implications [10]. An attribute implication is an expression $A \to B$ where A and B are subsets of attributes ($A, B \in 2^\mathcal{P}$) and it holds in a formal context if $A^\Delta \subseteq B^\Delta$ (equivalently $B \subseteq A^{\Delta\Delta}$). The semantics of the attribute implication is that, for every object $x \in \mathcal{O}$, if every attribute from A applies to the object x, then every attribute from B also applies to x. It is important to remark that the underlying semantics is a conjunctive one. Thus, our objective in the following is to consider additional knowledge in the form of so-called disjunctive attribute implications.

Table 1. Formal context $K_\mathcal{S}$.

$\mathcal{R}$	Man	Woman	Father	Mother	Parent
John	×				
Maria		×			
Peter	×		×		×
Clara		×		×	×

3 Asymmetric Composition of possibilistic operators

The Galois derivation operator which is at the basis of FCA theory is the operator of sufficiency $(.)^{\Delta}$. Some time ago, Dubois and Prade [6,9] have highlighted, in the setting of possibility theory, three other powerset derivation operators, namely the possibility operator (denoted $(.)^{\Pi}$), the necessity operator (denoted $(.)^{\mathrm{N}}$) and the dual sufficiency operator (denoted $(.)^{\nabla}$). The two former operators are given in the following:

— $(A)^{\Pi}$ corresponds to the set of objects that are associated with at least one attribute in A. Formally, we have:
$$(A)^{\Pi} = \{x \in \mathcal{O} \mid \exists a \in A, x\mathcal{R}a\}$$
— $(A)^{\mathrm{N}}$ corresponds to the set of objects such that any attribute that satisfies one of them is necessarily in A.
$$(A)^{\mathrm{N}} = \{x \in \mathcal{O} \mid \forall a \in \mathcal{P}\ (x\mathcal{R}a \Rightarrow a \in A)\}$$

$(X)^{\Pi}$ and $(X)^{\mathrm{N}}$ are dually obtained.

Let $x\overline{\mathcal{R}}a$ indicates that object x does not satisfy attribute a. In the particular case where the derivation operators $(.)^{\Pi}$, $(.)^{\mathrm{N}}$, $(.)^{\Delta}$ are applied to the complementary context $\overline{\mathcal{K}}(\mathcal{O},\mathcal{P},\overline{\mathcal{R}})$ (where $\overline{\mathcal{R}} = \{(x,a) \in \mathcal{O}\times\mathcal{P} \mid x\overline{\mathcal{R}}a\}$), we will exceptionally use the explicit notation $(.)^{\Pi}_{\overline{\mathcal{K}}}$, $(.)^{\mathrm{N}}_{\overline{\mathcal{K}}}$, $(.)^{\Delta}_{\overline{\mathcal{K}}}$. Given $X \subseteq \mathcal{O}$ and $\overline{X}$ its complementary set (i.e. $\mathcal{O} \setminus X$), the following recalls some useful properties [5] needed in the rest of the paper.

$$P_1 : X^{\Delta}_{\overline{\mathcal{K}}} = \overline{(X^{\Pi})}$$
$$P_2 : X^{\Delta}_{\overline{\mathcal{K}}} = (\overline{X})^{\mathrm{N}}$$
$$P_3 : X_1 \subseteq X_2 \Rightarrow (X_1)^{\Pi} \subseteq (X_2)^{\Pi}$$
$$P_4 : X \subseteq ((X)^{\Pi})^{\mathrm{N}}$$
$$P_5 : (X_1)^{\Pi} \cup (X_2)^{\Pi} = (X_1 \cup X_2)^{\Pi}$$
$$P_6 : X_1 \subseteq X_2 \Rightarrow (X_1)^{\mathrm{N}} \subseteq (X_2)^{\mathrm{N}}$$
$$P_7 : (X)^{\Pi} = (((X)^{\Pi})^{\mathrm{N}})^{\Pi}$$

These properties are dually satisfied for $A \subseteq \mathcal{P}$.

Let us also denote by NΠ-pair, a formal pair (X, A) s.t. $X = A^{\Pi}$ and $A = X^{\mathrm{N}}$, where X (resp. A) will be called NΠ-extent (resp. NΠ-intent). It may be remarked that both elements X and A present dual topological properties. Indeed, X is an open element, whereas A is a closed one, achieving then an "open-closed" pair. The set of all NΠ-pairs is denoted by $\mathfrak{B}_{\mathrm{N}\Pi}$, whereas the set $\mathfrak{B}_{\mathrm{N}\Pi}(Ext)$ (resp. $\mathfrak{B}_{\mathrm{N}\Pi}(Int)$) corresponds to the set of all NΠ-extents (resp. NΠ-intents). Proposition 1 establishes first a characterization of NΠ-pairs, whereas the proposition 2 gives the algebraic structure of the set $\mathfrak{B}_{\mathrm{N}\Pi}$.

Proposition 1. *Let $X \in 2^{\mathcal{O}}$ and $A \in 2^{\mathcal{P}}$, (X, A) is an* NΠ*-pair if and only if $(\overline{X}, A)$ is a formal concept in $\overline{\mathcal{K}}(\mathcal{O},\mathcal{P},\overline{\mathcal{R}})$.*

Proof. It is proved using properties P_1 and P_2 given in section 3.

It has been already established that the set $\mathfrak{B}_{\mathrm{N}\Pi}$ with a partial order (denoted $\preceq$) defined as $(X_1, A_1) \preceq (X_2, A_2)$ if $X_1 \subseteq X_2$ (or, equivalently, $A_1 \subseteq A_2$) forms a complete lattice, called the NΠ-lattice and denoted by $\mathfrak{L}_{\mathrm{N}\Pi}$. The following proposition gives the infima (greatest lower bound) and the suprema (least upper bound) for a given subset of $\mathfrak{L}_{\mathrm{N}\Pi}$.

Proposition 2. *The infima and suprema of a subset* (X_j, A_j) *(j an index set) of* $\mathfrak{L}_{N\Pi}$ *are given by:*

$$\bigwedge_{j\in J}(X_j, A_j) = (\bigcup_{j\in J} X_j, ((\bigcup_{j\in J} A_j)^{\Pi})^{N}); \qquad \bigvee_{j\in J}(X_j, A_j) = (\bigcap_{j\in J} X_j, \bigcap_{j\in J} A_j).$$

Proof. This result can be established using Proposition 1, and the fact that $(\overline{X}, A)$ is a formal concept of $\overline{\mathcal{K}}(\mathcal{O}, \mathcal{P}, \overline{\mathcal{R}})$.

Example 2. Figure 1 illustrates the $\mathfrak{L}_{N\Pi}$ lattice corresponding to the formal context given in Table 1.

Let us now introduce the mapping μ which associates to each set of attributes $A \in 2^{\mathcal{P}}$, its NΠ-pair such as:

$$\mu : 2^{\mathcal{P}} \rightarrow \mathcal{B}_{N\Pi}$$
$$\text{A} \rightarrow \mu(A) = (A^{\Pi}, (A^{\Pi})^{N})$$

The following proposition establishes the mapping μ for a set A of attributes.

Proposition 3. *Let* $A \subseteq \mathcal{P}$*, then* $\mu(A) = \bigwedge_{a\in A} \mu(\{a\})$

Proof. $A^{\Pi} = \bigcup_{a\in A} a^{\Pi}$ is obtained directly by the definition of possibility operator, we have $\mu(A) = (A^{\Pi}, (A^{\Pi})^{N}) \Leftrightarrow \mu(A) = (\bigcup_{a\in A} a^{\Pi}, (\bigcup_{a\in A} a^{\Pi})^{N}) = \bigwedge_{a\in A} \mu(\{a\})$

4 Disjunctive attribute implications

We propose now to introduce disjunctive attribute implications of the form $a_1 \vee \ldots \vee a_n \longmapsto b_1 \vee \ldots \vee b_m$ (equivalently denoted by $\bigvee A \longmapsto \bigvee B$ with $A = \{a_1, \ldots, a_n\}$,

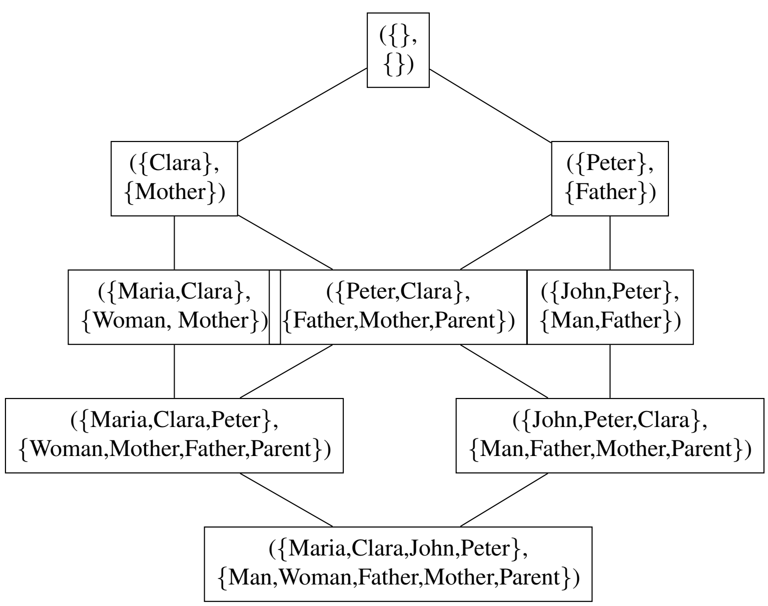

Fig. 1. Lattice $\mathfrak{L}_{N\Pi}$

and $B = \{b_1, ..., b_m\}$). Being understood that the satisfaction of such an implication is related to the set of all objects in O, we agree that a formal context $\mathcal{K}(O, \mathcal{P}, \mathcal{R})$ satisfies a disjunctive attribute implication $\bigvee A \longmapsto \bigvee B$ if and only if every object that is never satisfied by each attribute from B is also never satisfied by each attribute from A. Formally, $\mathcal{K} \models \bigvee A \longmapsto \bigvee B$, iff $\forall x \in O, if\ b_1 \nsubseteq \{x\}^{\Pi} \wedge ... \wedge b_m \nsubseteq \{x\}^{\Pi}\ then\ a_1 \nsubseteq \{x\}^{\Pi} \wedge ... \wedge a_n \nsubseteq \{x\}^{\Pi}$

For example, the formal context $K_{\mathcal{S}}$ given in Table 1 satisfies the disjunctive attribute implication Parent $\longmapsto$ Father $\vee$ Mother ($K_{\mathcal{S}} \models$ Parent $\longmapsto$ Father $\vee$ Mother).

The following important result can be easily obtained.

Proposition 4. *The disjunctive attribute implication $\bigvee A \longmapsto \bigvee B$ is valid in formal context $\mathcal{K}(O, \mathcal{P}, \mathcal{R})$ iff the attribute implication $B \longmapsto A$ is valid in formal context $\overline{\mathcal{K}}(O, \mathcal{P}, \overline{\mathcal{R}})$ iff $A \subseteq ((B)^{\Pi}_{\mathcal{K}})^{\mathrm{N}}_{\mathcal{K}}$.*

Proof. Suppose $B \longmapsto A$ is valid in $\overline{\mathcal{K}}$. In logical terms, it means $\wedge_{b \in B} \neg b \rightarrow \wedge_{a \in A} \neg a$, which is logically equivalent to $\vee_{a \in A} a \rightarrow \vee_{b \in B} b$. Now, $B \longmapsto A$ is valid in $\overline{\mathcal{K}}$ means $A \subseteq B^{\Delta\Delta}_{\overline{\mathcal{K}}}$, that is, $A \subseteq (\overline{B^{\Pi}_{\mathcal{K}}})^{\Delta}_{\overline{\mathcal{K}}}$ iff $A \subseteq ((B)^{\Pi}_{\mathcal{K}})^{\mathrm{N}}_{\mathcal{K}}$. □

A simpler way to assert the satisfaction of a disjunctive attribute implication based on the possibility operator $(.)^{\Pi}$ is given hereafter.

Proposition 5. *Given a formal context $\mathcal{K}(O, \mathcal{P}, \mathcal{R})$ and $A, B \subseteq \mathcal{P}$, $\mathcal{K} \models \bigvee A \longmapsto \bigvee B$ iff for each $x \in O$, $B \nsubseteq \overline{\{x\}^{\Pi}}$ or $A \subseteq \overline{\{x\}^{\Pi}}$.*

The disjunctive attribute implications that hold in a formal context $\mathcal{K}(O, \mathcal{P}, \mathcal{R})$ can be obtained from concept lattice $\mathfrak{L}_{\mathrm{N}\Pi}$. The following proposition illustrates this.

Proposition 6. *Given a formal context $\mathcal{K}(O, \mathcal{P}, \mathcal{R})$, $\mathcal{K} \models a \rightarrow \bigvee B$ iff $(a^{\Pi}, (a^{\Pi})^{\mathrm{N}}) \leq (B^{\Pi}, (B^{\Pi})^{\mathrm{N}})$*

This means that we have to check in the concept lattice $\mathfrak{L}_{\mathrm{N}\Pi}$ whether the NΠ-pairs associated to a are located above the infima of all NΠ-pairs associated to b from B.

Example 3. In the following we give the set of disjunctive attribute implications that matches to the formal context given in Table 1 by applying the proposition:
{Father→Man, Mother→Woman, Father∨Mother→Parent, Parent→Father∨Mother}

5 Possible and certain implications in incomplete contexts

The case of incomplete context has been only considered by Obiedkov [13] and by Burmeister and Holzer [2] until now. They have proposed to introduce a third value, denoted "?", in a formal context, which leads to the concept of an incomplete context, sometimes also called three values context. More formally, incomplete context $\mathcal{K}_i(O, \mathcal{P}, \{+, -, ?\}, \mathcal{R}_i)$ where O is the set of objects, $\mathcal{P}$ the set of attributes, "+", "-", "?" are the three possible entries of the incomplete context, and $\mathcal{R}$ is a ternary relation $\mathcal{R} \subseteq O \times \mathcal{P} \times \{+, -, ?\}$. The interpretation of the relation $\mathcal{R}$ is as follows. Let $x \in O$ and $a \in \mathcal{P}$:

— $(x, a, +) \in \mathcal{R}$: it is known that the object x has the attribute a
— $(x, a, -) \in \mathcal{R}$: it is known that the object x does not have the attribute a
— $(x, a, ?) \in \mathcal{R}$: it is unknown, whether the object x has the attribute a or not

An incomplete formal context may be viewed as a weighted family of all standard formal contexts obtained by changing unknown entries $(x, a, ?)$ into known ones ($(x, a, +)$ or $(x, a, -)$). The two extreme cases where all such unknown entries $(x, a, ?)$ are changed into $(x, a, -)$ and the case where all such unknown entries $(x, a, ?)$ are changed into $(x, a, +)$ give birth to lower and upper completions, respectively [8] [4].

In this way, two classical (Boolean) formal contexts, denoted $K_*(\mathcal{O}, \mathcal{P}, \mathcal{R}_*)$ and $K^*(\mathcal{O}, \mathcal{P}, \mathcal{R}^*)$ are obtained as respective results of the two replacements. More formally:

— $K_*(\mathcal{O}, \mathcal{P}, \mathcal{R}_*)$ is a Boolean formal context such that $\mathcal{R}_* = \{(x, a) | (x, a, +) \in \mathcal{R}_i\}$ where $A^{\Delta}_{K_*} = \{x | A \subseteq x\mathcal{R}_*\}$ is the set of objects certainly having all attributes in A
— $K^*(\mathcal{O}, \mathcal{P}, \mathcal{R}^*)$ is a Boolean formal context such that $\mathcal{R}^* = \{(x, a) | (x, a, +) \in \mathcal{R}_i$ or $(x, a, ?) \in \mathcal{R}_i\}$ where $A^{\Delta}_{K^*} = \{x | A \subseteq x\mathcal{R}^*\}$ is the set of objects possibly having all attributes in A.

There exists other intermediate formal contexts by replacing each "?" by "+" or "-" and we obtain exactly 2^n possible formal contexts (n is the number of "?" in the initial formal context). All attribute implications that are obtained from these formal contexts are either possible attribute implications or certain attribute implications. An implication is certain if it is valid in each formal context $\mathcal{K}_j$; this condition may seem hard to verify at first glance. The following theorem solves the problem.

Theorem 1. $A \longmapsto B$ is a certain attribute implication in $\mathcal{K}_i$ iff $A^{\Delta}_{K^*} \subseteq B^{\Delta}_{K_*}$

Proof. Assume that $A \longmapsto B$ is not a certain attribute implication in $\mathcal{K}_i$ and $A^{\Delta}_{K^*} \subseteq B^{\Delta}_{K_*}$. But $A \longmapsto B$ is not certain implication $\Longrightarrow \exists$ a formal context $\mathcal{K}_j | x \in A^{\Delta}_{K_j}$ and $x \notin B^{\Delta}_{K_j}$ $\Longrightarrow \exists$ an object x possibly having all attributes in A and not having the certain attributes in $B \Longrightarrow \exists x \in \mathcal{O} \,| x \in A^{\Delta}_{K^*}$ and $x \notin B^{\Delta}_{K_*} \Longrightarrow A^{\Delta}_{K^*} \nsubseteq B^{\Delta}_{K_*}$. □

Another problem is to determine a possible attribute implication that are holds in at least ont formal context $\mathcal{K}_j$, the following theorem facilitates this determination. Proofs are omitted due to space limitations.

Theorem 2. $A \longmapsto B$ is a possible attribute implication in $\mathcal{K}_i$ iff $A^{\Delta}_{K_*} \subseteq B^{\Delta}_{K^*}$

This section also considers disjunctive attribute implications, presented in section 4, in incomplete formal context $\mathcal{K}_i$. As in the case of conjunctive attribute implications we distinguish certain disjunctive attribute implications and possible disjunctive attribute implications. Note that $(A)^{\Pi}_{K_*}$ is the set of objects certainly having at least one attribute in A and $(A)^{\Pi}_{K^*}$ is the set of objects possibly having at least one attribute in A. And $\overline{(A)^{\Pi}_{K_*}}$ is the set of objects that certainly never have any attribute in A and $\overline{(A)^{\Pi}_{K^*}}$ is the set of objects that can never have any attribute in A. We get two major results of this paper.

Theorem 3. $\bigvee A \longmapsto \bigvee B$ is a certain disjunctive attribute implication in $\mathcal{K}_i$ iff $A^{\Pi}_{K^*} \subseteq B^{\Pi}_{K_*}$

Theorem 4. $\bigvee A \longmapsto \bigvee B$ is a possible disjunctive attribute implication in $\mathcal{K}_i$ iff $A^{\Pi}_{K_*} \subseteq B^{\Pi}_{K^*}$

6 Implications from gradually uncertain contexts

In an uncertain formal context the boxes are filled with a pair (α,β) of degree of necessity. That is to say that (α) is the necessity that the object has the attribute, and (β) is the necessity that the object does not have the attribute. Moreover, we should respect the property $\min(\alpha,\beta) = 0$ [7]. Pairs (1,0) and (0,1) correspond to completely informed situations where it is known that object has the attribute (ie. +), respectively the object does not have the attribute (ie. -). The pair (0,0) reflects total ignorance (ie. ?), whereas pairs (α,β) s.t. $1 > \max(\alpha,\beta) > 0$ correspond to partial ignorance.

Consider a pair of thresholds (u, v) with $u > 0$ and $v > 0$. $\mathcal{K}_{(u,v)}$ is an incomplete formal context obtained by replacing:

all entries of the form $(\alpha, 0)$ such that $\alpha \geq u$ by (+)
all entries of the form $(\alpha, 0)$ such that $\alpha < u$ by (?)
all entries of the form $(0,\beta)$ such that $\beta \geq v$ by (-)
all entries of the form $(0,\beta)$ such that $\beta < v$ by (?)

The classical formal context $(\mathcal{K}_{(u,v)})_*$ is obtained by replacing with (+) the pairs $(\alpha, 0)$ such that $\alpha \geq u$ and all the rest with (-). The classical formal context $(\mathcal{K}_{(u,v)})^*$ is obtained by replacing with (-) the pairs $(0,\beta)$ such that $\beta \geq v$ and all the rest with (+).

Observe that $(\mathcal{K}_{(u,v)})_*$ does not depend on v, and increases when u decreases. $(\mathcal{K}_{(u,v)})^*$ does not depend on u, and increases when v increases. Recall that $A^{\Delta}_{\mathcal{K}}$ increases as $\mathcal{K}$ increases (in the sense of inclusion). Therefore, $A^{\Delta}_{(\mathcal{K}_{(u,v)})^*}$ increases when v increases. $B^{\Delta}_{(\mathcal{K}_{(u,v)})_*}$ decreases when u increases.

An attribute implication $A \longmapsto B$ is more certain with u great and v great such that $A^{\Delta}_{(\mathcal{K}_{(u,v)})^*} \subseteq B^{\Delta}_{(\mathcal{K}_{(u,v)})_*}$. Therefore, the degree of certainty $\mathrm{cert}(A \longmapsto B)$ of the attribute implication is equal to the maximum value w such that $A^{\Delta}_{(\mathcal{K}_{(w,w)})^*} \subseteq B^{\Delta}_{(\mathcal{K}_{(w,w)})_*}$. In particular, $\mathrm{cert}(A \longmapsto B) = 1$ iff $A^{\Delta}_{(\mathcal{K}_{(1,1)})^*} \subseteq B^{\Delta}_{(\mathcal{K}_{(1,1)})_*}$ that is to say that the certain attribute implications are calculated with the most certain part of the data. Also a possibility degree is attached to the attribute implication such that $A^{\Delta}_{(\mathcal{K}_{(u,v)})_*} \subseteq B^{\Delta}_{(\mathcal{K}_{(u,v)})^*}$ which is all the greater as u and v are greater.

We also consider the disjunctive attribute implications in the uncertain formal context. Observe that $\overline{(\mathcal{K}_{(u,v)})_*}$ does not depend on v, and increases when u increases, and $\overline{(\mathcal{K}_{(u,v)})^*}$ does not depend on u, and increases when v decreases. Recall that the disjunctive attribute implication $\bigvee A \longmapsto \bigvee B$ is valid in a formal context $\mathcal{K}$ if and only if the attribute implication $B \longmapsto A$ is valid in $\overline{\mathcal{K}}$. Therefore, the degree of certainty $\mathrm{cert}(B \longmapsto A)$ is equal to the maximum value w such that $B^{\Delta}_{\overline{(\mathcal{K}_{(u,v)})_*}} \subseteq A^{\Delta}_{\overline{(\mathcal{K}_{(u,v)})^*}}$, equivalent to $\overline{B^{\Pi}_{(\mathcal{K}_{(u,v)})_*}} \subseteq \overline{A^{\Pi}_{(\mathcal{K}_{(u,v)})^*}}$, which is equivalently written: $A^{\Pi}_{(\mathcal{K}_{(u,v)})^*} \subseteq B^{\Pi}_{(\mathcal{K}_{(u,v)})_*}$. Also a possibility degree is attached to attribute implication such that $A^{\Pi}_{(\mathcal{K}_{(u,v)})^*} \subseteq B^{\Pi}_{(\mathcal{K}_{(u,v)})_*}$ which is all the greater as u and v are greater.

7 Conclusion

All existing works and approaches pertaining to FCA rely on the use of the classical Galois derivation operator (i.e. sufficiency operator). Thus, these works are based on the complete lattice of all formal concepts obtained using the composition of sufficiency operators. Consequently, induced implications are limited to their conjunctive form. In this paper we propose an approach that enlarges knowledge representation ability to disjunctive attribute implications. Possible links with [12] are to be investigated. The proposed approach considers "open-closed" pairs obtained by means of the asymmetric composition ($N \circ \Pi$) of necessity and possibility operators. We have only focused on composition $(.)^{N\Pi}$. Further researches should concern the study of other possible compositions of possibilistic composite operators such that $(.)^{\Pi\Delta}$, $(.)^{\nabla\Delta}$, etc.

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