Extracting $\mathcal{ALEQR}^{\mathsf{Self}}$-Knowledge Bases from Graphs

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Abstract. A description graph is a directed graph that has labeled vertices and edges. This document proposes a method for extracting a knowledge base from a description graph. The technique is presented for the description logic $\mathcal{ALEQR}^{\mathsf{Self}}$, which allows for conjunctions, primitive negations, existential restrictions, value restrictions, qualified number restrictions, existential self restrictions, general concept inclusions, and complex role inclusions. Furthermore, also sublogics may be chosen to express the axioms in the knowledge base. The extracted knowledge base entails exactly all those statements that can be expressed in the chosen description logic and are encoded in the input graph.

Keywords: Description Logics · Formal Concept Analysis · Terminological Learning · Knowledge Base · General Concept Inclusion · Canonical Base · Most-Specific Concept Description · Interpretation · Description Graph · Folksonomy · Social Network

1 Introduction

There have been several approaches towards the combination of description logics [5] and formal concept analysis [12] for knowledge acquisition, knowledge exploration, and knowledge completion. Rudolph [19] invented a method for the exploration of concept inclusions holding in an $\mathcal{FLE}$-interpretation. Baader, Ganter, Sattler, and Sertkaya, [3, 4, 20] provided a technique for completion of knowledge bases. Furthermore, Baader and Distel [1, 2, 9] gave a method for computing a finite base of all concept inclusions holding in a finite $\mathcal{EL}$-interpretation by means of the Duquenne-Guiges-base [13] of a so-called induced formal context. Finally, Borchmann [6–8] extended the results by defining the notion of confidence for concept inclusions, and utilized the Luxenburger-base [16–18] of the induced formal context to formulate a base for the concept inclusions whose confidence exceeds a given threshold.

In the following text we provide a method to compute a knowledge base for concept and role inclusions holding in an $\mathcal{ALEQR}^{\mathsf{Self}}$-interpretation or description graph, respectively, which entails all knowledge that is encoded in the interpretation/graph and can be expressed in $\mathcal{ALEQR}^{\mathsf{Self}}$. For this purpose we need the notion of a model-based most-specific concept description. It is defined as a concept description which describes a given individual x, i.e., the individual is an instance of the concept, and is most specific w.r.t. this property, i.e., for all concept descriptions C that have x as an individual, the most-specific concept is subsumed by C. Since we do not want to use greatest fixpoint semantics here, we restrict the role-depth to ensure existence of most-specific concepts. The description logic $\mathcal{ALEQR}^{\mathsf{Self}}$ is chosen for knowledge representation here, since it is an expressive description logic that does not allow for disjunctions (like $\mathcal{ALC}$), and hence will not model the examples in the input graph too exactly.

We start with a short introduction of the description logic $\mathcal{ALEQR}^{\mathsf{Self}}$. Then we define description graphs, and show their equivalence to interpretations. Furthermore, we then present model-based most-specific concept descriptions, and their relationships to formal concept analysis. We then continue with induced concept contexts and induced role contexts, and eventually utilize them to construct the desired knowledge base.

Please note that many of the results on model-based most-specific concept descriptions, induced concept contexts, and bases of general concept inclusions, have already been observed and proven by Baader and Distel [1, 2, 9] for the light-weight description logic $\mathcal{EL}^{\bot}$ w.r.t. greatest fixpoint semantics, which allows for the bottom concept, conjunctions, existential restrictions, and general concept inclusions. Their results are extended to the additional concept constructors of $\mathcal{ALEQR}^{\mathsf{Self}}$, and furthermore we also take complex role inclusions into account.

2 The Description Logic $\mathcal{ALEQR}^{\mathsf{Self}}$

Let (N_C, N_R) be a *signature*, i.e., N_C is a set of *concept names*, and N_R is a set of *role names*, such that N_C and N_R are disjoint. We stick to the usual notations and hence concept names are written as upper-case latin letters, e.g., A and B, and role names are written as lower-case latin letters, e.g., r and s. An *interpretation* over (N_C, N_R) is a tuple $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ where $\Delta^{\mathcal{I}}$ is a non-empty set, called *domain*, and $\cdot^{\mathcal{I}}$ is an *extension function* that maps concept names $A \in N_C$ to subsets $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ and role names $r \in N_R$ to binary relations $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$.

The set of all $\mathcal{ALEQR}^{\mathsf{Self}}$-*concept descriptions* is denoted by $\mathcal{ALEQR}^{\mathsf{Self}}(N_C, N_R)$, and is inductively defined as follows. Every concept name $A \in N_C$, the *bottom concept* $\bot$, and the *top concept* $\top$, is an *atomic* $\mathcal{ALEQR}^{\mathsf{Self}}$-concept description. If $A \in N_C$ is a concept name, $r \in N_R$ is a role name, $C, D \in \mathcal{ALEQR}^{\mathsf{Self}}(N_C, N_R)$ are concept descriptions, and $n \in \mathbb{N}_+$ is a positive integer, then $\neg A$, $C \sqcap D$, $\exists\, r.\, C$, $\forall\, r.\, C$, $\geq n.\, r.\, C$, $\leq n.\, r.\, C$, and $\exists\, r.\, \mathsf{Self}$, are *complex* $\mathcal{ALEQR}^{\mathsf{Self}}$-concept descriptions. The extension function of an interpretation $\mathcal{I}$ is canonically extended to all $\mathcal{ALEQR}^{\mathsf{Self}}$-concept descriptions as shown in the semantics column of Figure 1.

Note that every individual without any r-successors in the interpretation $\mathcal{I}$ at all is an element of the extension of every value restriction $\forall\, r.\, C$ for arbitrary concept descriptions C. We use the usual notation $\binom{X}{k}$ for the set of all subsets of X with exactly k elements. It is well-known that $\left|\binom{X}{k}\right| = \binom{|X|}{k}$.

Furthermore, $\mathcal{ALEQR}^{\mathsf{Self}}$ allows to express the following *terminological axioms*. If A is a concept name, and C, D are concept descriptions, then $C \sqsubseteq D$ is a *(general) concept inclusion* (abbr. *GCI*), and $A \equiv C$ is a *concept definition*. Of course, every concept definition $A \equiv C$ can be simulated by two concept inclusions $A \sqsubseteq C$ and $C \sqsubseteq A$. If $r, r_1, \ldots, r_n, s$ are role names, then $r \sqsubseteq s$ is a *simple role inclusion*, and $r_1 \circ \ldots \circ r_n \sqsubseteq s$ is a *complex role inclusion*, also called *role inclusion axiom* (abbr. *RIA*). We then say that an interpretation $\mathcal{I}$ is a *model* of an axiom α, denoted as $\mathcal{I} \models \alpha$, if the condition in the

name	*syntax C*	*semantics* $C^{\mathcal{I}}$
bottom concept	$\bot$	$\emptyset$
top concept	$\top$	$\Delta^{\mathcal{I}}$
primitive negation	$\neg A$	$\Delta^{\mathcal{I}} \setminus A^{\mathcal{I}}$
conjunction	$C \sqcap D$	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$
existential restriction	$\exists r.C$	$\{ x \in \Delta^{\mathcal{I}} \mid \exists y \in \Delta^{\mathcal{I}} \colon (x,y) \in r^{\mathcal{I}} \land y \in C^{\mathcal{I}} \}$
value restriction	$\forall r.C$	$\{ x \in \Delta^{\mathcal{I}} \mid \forall y \in \Delta^{\mathcal{I}} \colon (x,y) \in r^{\mathcal{I}} \to y \in C^{\mathcal{I}} \}$
qualified number	$\geq n.r.C$	$\{ x \in \Delta^{\mathcal{I}} \mid \exists Y \in \binom{\Delta^{\mathcal{I}}}{n} \colon \{ x \} \times Y \subseteq r^{\mathcal{I}} \land Y \subseteq C^{\mathcal{I}} \}$
restriction	$\leq n.r.C$	$\{ x \in \Delta^{\mathcal{I}} \mid \forall Y \in \binom{\Delta^{\mathcal{I}}}{n+1} \colon \{ x \} \times Y \subseteq r^{\mathcal{I}} \to Y \not\subseteq C^{\mathcal{I}} \}$
self restriction	$\exists r.\mathsf{Self}$	$\{ x \in \Delta^{\mathcal{I}} \mid (x,x) \in r^{\mathcal{I}} \}$

Fig. 1. Concept Constructors of $\mathcal{ALEQR}^{\mathsf{Self}}$

semantics column of Figure 2 is satisfied. An axiom is *generally valid* if all interpretations are models of it. If $C \sqsubseteq D$ is generally valid, then we denote this by $C \sqsubseteq D$, too, and say that C is *subsumed* by D, C is a *subsumee* of D, and D is a *subsumer* of C.

A *TBox* is a set of concept inclusions and concept definitions, and a *RBox* is a set of role inclusions. $\mathcal{I}$ is a *model* of a TBox $\mathcal{T}$, denoted as $\mathcal{I} \models \mathcal{T}$, if $\mathcal{I}$ is a model of all axioms $\alpha \in \mathcal{T}$, and analogously for RBoxes $\mathcal{R}$. A *knowledge base* $\mathcal{K}$ is a pair $(\mathcal{T}, \mathcal{R})$ of a TBox $\mathcal{T}$ and a RBox $\mathcal{R}$.

name	*syntax* α	*semantics* $\mathcal{I} \models \alpha$
concept inclusion	$C \sqsubseteq D$	$C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$
concept definition	$A \equiv C$	$A^{\mathcal{I}} = C^{\mathcal{I}}$
simple role inclusion	$r \sqsubseteq s$	$r^{\mathcal{I}} \subseteq s^{\mathcal{I}}$
complex role inclusion	$r_1 \circ r_2 \circ \ldots \circ r_n \sqsubseteq s$	$r_1^{\mathcal{I}} \circ r_2^{\mathcal{I}} \circ \ldots \circ r_n^{\mathcal{I}} \subseteq s^{\mathcal{I}}$

Fig. 2. Axiom Constructors of $\mathcal{ALEQR}^{\mathsf{Self}}$

$\circ$ denotes the *product* operator where $R \circ S := \{ (x,z) \mid \exists y \colon (x,y) \in R \land (y,z) \in S \}$.

Definition 1 (Knowledge Base). *Let $\mathcal{I}$ be an interpretation. A* knowledge base *for $\mathcal{I}$ is a knowledge base $\mathcal{K}$ that has the following properties.*

(sound) *All axioms in $\mathcal{K}$ hold in $\mathcal{I}$, i.e., $\mathcal{I} \models \mathcal{K}$.*
(complete) *All axioms that hold in $\mathcal{I}$, are entailed by $\mathcal{K}$, i.e., $\mathcal{I} \models \alpha \Rightarrow \mathcal{K} \models \alpha$.*
(irredundant) *None of the axioms in $\mathcal{K}$ follows from the others, i.e., $\mathcal{K} \setminus \{\alpha\} \not\models \alpha$ for all $\alpha \in \mathcal{K}$.*

3 Graphs

The semantics of $\mathcal{ALEQR}^{\mathsf{Self}}$ can also be characterized by means of description graphs, which are cryptomorphic to interpretations. A *description graph* over (N_C, N_R) is a tuple $\mathcal{G} = (V, E, \ell)$, such that the following conditions hold.

1. (V, E) is a directed graph, i.e., V is a set of *vertices*, and $E \subseteq V \times V$ is a set of *directed edges* on V. For an edge $(v, w) \in E$ we say that v and w are connected, v is the *source vertex*, and w is the *target vertex* of (v, w).
2. $\ell = \ell_V \dot\cup \ell_E$ is a *labeling function* where $\ell_V \colon V \to 2^{N_C}$ maps each vertex $v \in V$ to a label set $\ell_V(v) \subseteq N_C$, and $\ell_E \colon E \to 2^{N_R}$ maps each edge $(v, w) \in E$ to a label set $\ell_E(v, w) \subseteq N_R$.

The vertices of the graph $\mathcal{G}$ are labeled with subsets of N_C to indicate the concept names they belong to. Analogously, the edges are labeled with subsets of N_R to allow multiple (named) relations between the same two vertices in the graph. Usually, one would also specify a root vertex $v_0 \in V$ for description graphs, but this is not necessary for our purposes here.

A description graph may also be called *folksonomy* or *social network* here. For example, the set N_R of role names in the signature may contain a relation friend that connects friends in a social network (graph). Other relations are for example isMarriedWith, sentFriendrequestTo, likes, follows, and hasAttendedEvent, with their obvious meaning. The vertices in a social network are of course the users (and possibly other objects). The vertex labels in the set N_C of concept names can be used to categorize the users in a social network, e.g., by nationality, sex, marital status, profession, etc.

For each description graph $\mathcal{G} = (V, E, \ell)$ we define a canonical interpretation $\mathcal{I}_\mathcal{G}$ that contains all information that is provided in $\mathcal{G}$ as follows. The domain is just the vertex set, i.e., $\Delta^{\mathcal{I}_\mathcal{G}} := V$, and the extensions of concept names $A \in N_C$, and of role names $r \in N_R$, respectively, are given as follows.

$$A^{\mathcal{I}_\mathcal{G}} := \ell_V^{-1}(A) = \{\, v \in V \mid A \in \ell(v) \,\}$$
$$r^{\mathcal{I}_\mathcal{G}} := \ell_E^{-1}(r) = \{\, (v, w) \in E \mid r \in \ell(v, w) \,\}$$

Furthermore, we can easily construct a description graph $\mathcal{G}_\mathcal{I}$ from an interpretation $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ by setting $\mathcal{G}_\mathcal{I} := (V, E, \ell)$ where

$$V := \Delta^\mathcal{I} \qquad \ell_V(v) := \left\{\, A \in N_C \;\middle|\; v \in A^\mathcal{I} \,\right\}$$
$$E := \bigcup_{r \in N_R} r^\mathcal{I} \qquad \ell_E(v, w) := \left\{\, r \in N_R \;\middle|\; (v, w) \in r^\mathcal{I} \,\right\}.$$

It can be readily verified that both transformations are mutually inverse, i.e., $\mathcal{I}_{\mathcal{G}_\mathcal{I}} = \mathcal{I}$ for all interpretations $\mathcal{I}$, and $\mathcal{G}_{\mathcal{I}_\mathcal{G}} = \mathcal{G}$ for all description graphs $\mathcal{G}$.

As a consequence, we do not have to distinguish between interpretations and description graphs, and we may also compute model-based most-specific concept descriptions (which are usually defined for individuals of an interpretation, cf. next section) for vertices in description graphs. In the following we want to propose a method to compute a knowledge base $\mathcal{K} = (\mathcal{T}, \mathcal{R})$ from a given description graph $\mathcal{G}$ that entails all knowledge that is encoded in $\mathcal{G}$ and is expressible in the description logic $\mathcal{ALEQR}^{\mathsf{Self}}$.

4 Model-Based Most-Specific Concept Descriptions

The *role depth* $\mathsf{rd}(C)$ of a concept description C is defined as the greatest number of roles in a path in the syntax tree of C. Formally, we inductively define the role depth as follows.

1. Every atomic concept description $A, \bot, \top$, and every primitive negation $\neg A$, has role depth 0.
2. The role depth of a conjunction is the maximum of the role depths of the conjuncts, i.e., $\mathsf{rd}(C \sqcap D) := \mathsf{rd}(C) \vee \mathsf{rd}(D)$ for all concept descriptions C and D.
3. The role depth of a restriction is the successor of the role depth of the concept description in the restriction's body, i.e., $\mathsf{rd}(Q\, r.\, C) := 1 + \mathsf{rd}(C)$ for all quantifiers $Q \in \{\exists, \forall, \geq n, \leq n\}$, role names $r \in N_R$, and concept descriptions C.
4. The role depth of a self restriction is just defined as 1, i.e., $\mathsf{rd}(\exists\, r.\, \mathsf{Self}) := 1$.

It is easy to see that the role-depth of a concept description is well-defined. However, equivalent concept descriptions do not necessarily have the same role depth. For example the concept description $\bot$ and $\exists r.\bot$ are equivalent, but the former concept description has role depth 0 and the latter has role depth 1.

Definition 2 (Model-Based Most-Specific Concept Description). *Let (N_C, N_R) be a signature, $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ an interpretation over (N_C, N_R), $\delta \in \mathbb{N}$ a role-depth bound, and $X \subseteq \Delta^{\mathcal{I}}$ a subset of the interpretation's domain. Then an $\mathcal{ALEQR}^{\mathsf{Self}}$-concept description C is called a* model-based most-specific concept description *(abbr.* mmsc*) of X w.r.t. $\mathcal{I}$ and δ if it satisfies the following conditions.*

1. *C has a role depth of at most d, i.e., $\mathsf{rd}(C) \leq \delta$.*
2. *All elements of X are in the extension of C w.r.t. $\mathcal{I}$, i.e., $X \subseteq C^{\mathcal{I}}$.*
3. *For all concept descriptions D with $\mathsf{rd}(D) \leq \delta$ and $X \subseteq D^{\mathcal{I}}$ it holds that $C \sqsubseteq D$.*

Since all model-based most-specific concept descriptions of X w.r.t. $\mathcal{I}$ and δ are unique up to equivalence, we speak of *the* mmsc, and denote it by $X^{\mathcal{I}_\delta}$.

Lemma 3. *Let $\mathcal{I}$ be an interpretation over the signature (N_C, N_R). Then the following statements hold for all subsets $X, Y \subseteq \Delta^{\mathcal{I}}$, and concept descriptions $C, D \in \mathcal{ALEQR}^{\mathsf{Self}}(N_C, N_R)$ with a role-depth $\leq \delta$.*

1. *$X \subseteq C^{\mathcal{I}}$ if, and only if, $X^{\mathcal{I}_\delta} \sqsubseteq C$.*
2. *$X \subseteq Y$ implies $X^{\mathcal{I}_\delta} \sqsupseteq Y^{\mathcal{I}_\delta}$.*
3. *$C \sqsubseteq D$ implies $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.*
4. *$X \subseteq X^{\mathcal{I}_\delta \mathcal{I}}$.*
5. *$C \sqsupseteq C^{\mathcal{I}\mathcal{I}_\delta}$.*
6. *$X^{\mathcal{I}_\delta} \equiv X^{\mathcal{I}_\delta \mathcal{I} \mathcal{I}_\delta}$.*
7. *$C^{\mathcal{I}} = C^{\mathcal{I}\mathcal{I}_\delta \mathcal{I}}$.*

It then follows that $\cdot^{\mathcal{I}\mathcal{I}_\delta}$ is a closure operator on the concept description poset $(\mathcal{ALEQR}^{\mathsf{Self}}(N_C, N_R), \sqsupseteq)$ factorized by concept equivalence, and a concept inclusion $C \sqsubseteq D$ holds in $\mathcal{I}$ if, and only if, the implication $C \to D$ holds in the closure operator $\cdot^{\mathcal{I}\mathcal{I}_\delta}$. It follows that there is a (finite) canonical base of concept inclusions holding in a (finite) interpretation $\mathcal{I}$.

Definition 4 (Least Common Subsumer). *Let C, D be $\mathcal{ALEQR}^{\mathsf{Self}}$-concept descriptions w.r.t. the signature (N_C, N_R). Then a concept description $E \in \mathcal{ALEQR}^{\mathsf{Self}}(N_C, N_R)$ is called a* least common subsumer *(abbr.* lcs*) of C and D if the following conditions are fulfilled.*

1. *E subsumes both C and D, i.e., $C \sqsubseteq E$ and $D \sqsubseteq E$.*
2. *Whenever F is a common subsumer of C and D, then F subsumes E, i.e., $C \sqsubseteq F$ and $D \sqsubseteq F$ implies $E \sqsubseteq F$ for all concept descriptions $F \in \mathcal{ALEQR}^{\mathsf{Self}}(N_C, N_R)$.*

It follows that least common subsumers are always unique up to equivalence. Hence, we can speak of *the* lcs of two concept descriptions, and furthermore we denote it by $\mathsf{lcs}(C, D)$ or $C \sqcup D$. The definition can be canonically extended to an arbitrary number of concept descriptions, and we then write $\mathsf{lcs}(C_1, \ldots, C_n)$ or $\bigsqcup_{i=1}^{n} C_i$ for the least common subsumer of the concept descriptions $C_1, \ldots, C_n$.

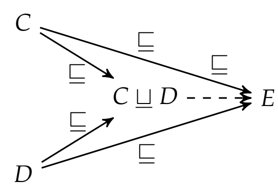

Fig. 3. The least common subsumer is a pullback in the category, whose objects are concept descriptions and whose morphisms are subsumptions.

Lemma 5. *Let $(X_t)_{t \in T}$ be a family of subsets $X_t \subseteq \Delta^{\mathcal{I}}$, and $(C_s)_{s \in S}$ a family of concept descriptions $C_s \in \mathcal{ALEQR}^{\mathsf{Self}}(N_C, N_R)$. Then the following statements hold.*

1. $(\bigcup_{t \in T} X_t)^{\mathcal{I}_\delta} \equiv \bigsqcup_{t \in T} X_t^{\mathcal{I}_\delta}$
2. $(\bigsqcap_{s \in S} C_s)^{\mathcal{I}} = \bigcap_{s \in S} C_s^{\mathcal{I}}$

Lemma 6. *If $C \sqsubseteq D$ holds in $\mathcal{I}$, and both C and D have a role depth $\leq \delta$, then also $C \sqsubseteq C^{\mathcal{I}\mathcal{I}_\delta}$ holds in $\mathcal{I}$, and $C \sqsubseteq D$ follows from $C \sqsubseteq C^{\mathcal{I}\mathcal{I}_\delta}$.*

Beforehand we have observed a pair of mappings that has similar properties like the well-known galois connection which is induced by a formal context. More specifically, the pair $(\cdot^{\mathcal{I}_\delta}, \cdot^{\mathcal{I}})$ is an adjunction. Consequently, we adapt the notions of a formal concept and a formal concept lattice as follows.

Definition 7 (Description Concept). *Let $\mathcal{I}$ be a finite interpretation over the signature (N_C, N_R), and $\delta \in \mathbb{N}$ a role-depth bound.*

A description concept *of $\mathcal{I}$ and δ is a pair (X, C) that consists of a subset $X \subseteq \Delta^{\mathcal{I}}$, and an $\mathcal{ALEQR}^{\mathsf{Self}}$-concept description over (N_C, N_R), such that X is the extension $C^{\mathcal{I}}$, and C is the model-based most-specific concept description $X^{\mathcal{I}_\delta}$. Furthermore, we call X the* extent*, and C the* intent *of (X, C). The set of all description concepts of $\mathcal{I}$ and δ is denoted as $\mathfrak{B}(\mathcal{I}, \delta)$. Analogously, $\mathsf{Ext}(\mathcal{I}, \delta)$ and $\mathsf{Mmsc}(\mathcal{I}, \delta)$ denote the sets of all extents and intents, respectively.*

To ensure formal correctness, we require that $\mathfrak{B}(\mathcal{I},\delta)$ only contains at most one description concept with the extent X. This is no limitation as we will see in the next lemma that all description concepts with the same extent have equivalent intents.

Definition 8 (Subconcept, Superconcept, Description Concept Lattice). *Let (X,C) and (Y,D) be two description concepts. Then (X,C) is a* subconcept *of (Y,D) if $X \subseteq Y$ holds. We then also write $(X,C) \leq (Y,D)$, and call (Y,D) a* superconcept *of (X,C).*

Additionally, the pair $\underline{\mathfrak{B}}(\mathcal{I},\delta) := (\mathfrak{B}(\mathcal{I},\delta),\leq)$ is called description concept lattice *of $\mathcal{I}$ and δ.*

Lemma 9 (Order on Description Concepts). *Let $\mathcal{I}$ be a finite interpretation over the signature (N_C,N_R), and $\delta \in \mathbb{N}$ a role-depth bound.*

1. *For two description concepts (X,C) and (Y,D) it is true that*

$$(X,C) \leq (Y,D) \Leftrightarrow X \subseteq Y \Leftrightarrow C \sqsubseteq D.$$

2. *The relation $\leq$ is an order on $\mathfrak{B}(\mathcal{I},\delta)$.*

We may furthermore observe that the set of all description concepts with the given order $\leq$ is a complete lattice.

Definition 10 (Description Lattice). *Let $\mathcal{I}$ be a finite interpretation over the signature (N_C,N_R), and $\delta \in \mathbb{N}$ a role-depth bound. Then $\underline{\mathfrak{B}}(\mathcal{I},\delta)$ is a complete lattice whose infima and suprema are given by the following equations.*

$$\bigwedge_{t\in T}(X_t,C_t) = \left(\bigcap_{t\in T} X_t, \left(\bigsqcap_{t\in T} C_t\right)^{\mathcal{I}\mathcal{I}_\delta}\right)$$

$$\bigvee_{t\in T}(X_t,C_t) = \left(\left(\bigcup_{t\in T} X_t\right)^{\mathcal{I}_\delta\mathcal{I}}, \underline{\bigsqcup_{t\in T}} C_t\right)$$

A description lattice is a nice visualization of the information provided in a description graph or in an interpretation, respectively. Since interpretations and description graphs are cryptomorphically defined, we do not need to further distinguish between them. One can think of description lattices as a natural generalization of concept lattices which do not only allow conjunctions of attributes as intents, but also more complex concept descriptions that can be expressed in the underlying description logic. Of course, if the chosen description logic is $\mathcal{L}_0$, i.e., only allows for conjunctions $\sqcap$, then the concept lattices and description lattices w.r.t. $\mathcal{L}_0$ coincide. However, for more complex description logics like $\mathcal{EL}$ or $\mathcal{FLE}$ or extensions thereof, we can further involve roles in the intents of the description concepts which adds further expressivity.

There is also a strong correspondence to the pattern structures and their lattices that have been introduced by Ganter and Kuznetsov [11]. Of course, the set of patterns consists of all concept descriptions that are expressible in the underlying description logic w.r.t. the given signature (N_C,N_R). The similarity operation is simply given by the least common subsumer mapping $\underline{\sqcup}$ which is the infimum in the lattice of all concept descriptions.

5 Induced Concept Contexts

Definition 11 (Induced Context). *Let $\mathcal{I}$ be an interpretation, and $\mathcal{M}$ a set of concept descriptions, both over the signature (N_C, N_R). Then the* induced context *of $\mathcal{I}$ and $\mathcal{M}$ is defined as the formal context $\mathbb{K}_{\mathcal{I},\mathcal{M}} := (\Delta^{\mathcal{I}}, \mathcal{M}, I)$, where the incidence I is defined via $(x, C) \in I$ if, and only, if $x \in C^{\mathcal{I}}$. For a concept description C over (N_C, N_R) its* projection *to $\mathcal{M}$ is defined as $\pi_{\mathcal{M}}(C) := \{ D \in \mathcal{M} \mid C \sqsubseteq D \}$. A concept description C is* expressible in terms of *$\mathcal{M}$ if $C \equiv \bigsqcap U$ for a subset $U \subseteq \mathcal{M}$. We have $\bigsqcap \emptyset = \top$ and $\bigsqcap U = \bigsqcap_{C \in U} C$ for all subsets $\emptyset \neq U \subseteq \mathcal{M}$.*

Lemma 12. *Let $\mathcal{I}$ be an interpretation, and $\mathcal{M}$ a set of concept descriptions. Then the following statements hold for all subsets $X, Y \subseteq \mathcal{M}$, and all concept descriptions C, D.*

1. *$X \subseteq \pi_{\mathcal{M}}(C)$ if, and only if, $C \sqsubseteq \bigsqcap X$.*
2. *$X \subseteq Y$ implies $\bigsqcap X \sqsupseteq \bigsqcap Y$.*
3. *$C \sqsubseteq D$ implies $\pi_{\mathcal{M}}(C) \supseteq \pi_{\mathcal{M}}(D)$.*
4. *$X \subseteq \pi_{\mathcal{M}}(\bigsqcap X)$*
5. *$C \sqsubseteq \bigsqcap \pi_{\mathcal{M}}(C)$.*
6. *$\bigsqcap X \equiv \bigsqcap \pi_{\mathcal{M}}(\bigsqcap X)$.*
7. *$\pi_{\mathcal{M}}(C) = \pi_{\mathcal{M}}(\bigsqcap \pi_{\mathcal{M}}(C))$.*

Lemma 13. *Let $\mathbb{K}_{\mathcal{I},\mathcal{M}}$ be an induced context. Then the following statements hold for all concept descriptions C over (N_C, N_R), all subsets $U \subseteq \mathcal{M}$, and $X \subseteq \Delta^{\mathcal{I}}$.*

1. *$\pi_{\mathcal{M}}(X^{\mathcal{I}}) = X^I$*
2. *$(\bigsqcap U)^{\mathcal{I}} = U^I$*
3. *$C^{\mathcal{I}} \subseteq \pi_{\mathcal{M}}(C)^I$*
4. *$\pi_{\mathcal{M}}\left((\bigsqcap U)^{\mathcal{I}\mathcal{I}}\right) = U^{II}$*
5. *$C \equiv \bigsqcap \pi_{\mathcal{M}}(C)$ if C is expressible in terms of $\mathcal{M}$.*
6. *$C^{\mathcal{I}} = \pi_{\mathcal{M}}(C)^I$ if C is expressible in terms of $\mathcal{M}$.*
7. *$U = \pi_{\mathcal{M}}(\bigsqcap U)$ if U is an intent of $\mathbb{K}_{\mathcal{I},\mathcal{M}}$.*

The next lemma tells us that we can directly decide in the induced context $\mathbb{K}_{\mathcal{I},\mathcal{M}}$, whether a concept inclusion between conjunctions of concept descriptions of $\mathcal{M}$ holds in the given interpretation $\mathcal{I}$.

Lemma 14 (Implications and concept inclusions). *Let $\mathcal{I}$ be an interpretation, and $\mathcal{M}$ a set of concept descriptions, both over the signature (N_C, N_R). Then for all subsets $X, Y \subseteq \mathcal{M}$, the concept inclusion $\bigsqcap X \sqsubseteq \bigsqcap Y$ holds in $\mathcal{I}$ if, and only if, the implication $X \to Y$ holds in $\mathbb{K}_{\mathcal{I},\mathcal{M}}$.*

Definition 15 (Approximation). *Let $\mathcal{I}$ be an interpretation over the signature (N_C, N_R), $\delta \in \mathbb{N}$ a role-depth bound, and $C \in \mathcal{ALEQR}^{\mathsf{Self}}(N_C, N_R)$ a concept description with its normal form $\bigsqcap_{A \in U} A \sqcap \bigsqcap_{(Q,r,D) \in \Pi} Q\, r.\, D$. Then the* approximation *of C w.r.t. $\mathcal{I}$ and δ is defined as the concept description*

$$\lfloor C \rfloor_{\mathcal{I},\delta} := \bigsqcap_{A \in U} A \sqcap \bigsqcap_{(Q,r,D) \in \Pi} Q\, r.\, D^{\mathcal{I}\mathcal{I}_\delta}.$$

Lemma 16. *For all concept descriptions C, D, and role names r, the following statements hold.*

1. $(C^{\mathcal{I}\mathcal{I}_\delta} \sqcap D)^{\mathcal{I}} = (C \sqcap D)^{\mathcal{I}}$.
2. $(Q\, r.\, C^{\mathcal{I}\mathcal{I}_\delta})^{\mathcal{I}} = (Q\, r.\, C)^{\mathcal{I}}$ *for all quantifiers* $Q \in \{\exists, \forall, \geq n, \leq n\}$.

Lemma 17. *For every interpretation* $\mathcal{I}$*, and every concept description* C *it holds that*

$$C^{\mathcal{I}\mathcal{I}_\delta} \sqsubseteq \lfloor C \rfloor_{\mathcal{I},\delta} \sqsubseteq C.$$

Lemma 18. *Let* $\mathcal{I}$ *be an interpretation, and* $\delta \in \mathbb{N}$ *a role-depth bound. Define*

$$\mathcal{M}_{\mathcal{I},\delta} := \{\, \bot \,\} \cup \{\, A, \neg A \mid A \in N_C \,\} \cup \left\{ \begin{array}{l|l} \exists\, r.\, X^{\mathcal{I}_{\delta-1}}, & \\ \forall\, r.\, X^{\mathcal{I}_{\delta-1}}, & r \in N_R, \\ \geq n.\, r.\, X^{\mathcal{I}_{\delta-1}}, & 1 \leq m < n \leq \left|\Delta^{\mathcal{I}}\right|, \\ \leq m.\, r.\, X^{\mathcal{I}_{\delta-1}}, & \emptyset \neq X \subseteq \Delta^{\mathcal{I}} \\ \exists\, r.\, \mathsf{Self} & \end{array} \right\}.$$

Then every model-based most specific concept description of $\mathcal{I}$ *with role-depth* $\leq \delta$ *is expressible in terms of* $\mathcal{M}_{\mathcal{I},\delta}$*. Furthermore, the* induced context *of* $\mathcal{I}$ *and* δ *is defined as the induced context* $\mathbb{K}^{\mathsf{C}}_{\mathcal{I},\delta} := \mathbb{K}_{\mathcal{I},\mathcal{M}_{\mathcal{I},\delta}}$ *of* $\mathcal{I}$ *and* $\mathcal{M}_{\mathcal{I},\delta}$*.*

Lemma 19 (Intents and MMSCs). *Let* $\mathcal{I}$ *be an interpretation over* (N_C, N_R)*, and* $\mathbb{K}_{\mathcal{I},\delta}$ *its induced context w.r.t. the role-depth bound* $\delta \in \mathbb{N}$*. Then the following statements hold for all subsets* $U \subseteq \mathcal{M}_{\mathcal{I},\delta}$*, and concept descriptions* C *over* (N_C, N_R)*.*

1. $(\bigsqcap U)^{\mathcal{I}\mathcal{I}_\delta} \equiv \bigsqcap U^{II}$.
2. *If* U *is an intent of* $\mathbb{K}_{\mathcal{I},\delta}$*, then* $\bigsqcap U$ *is a mmsc of* $\mathcal{I}$ *with role-depth* $\leq \delta$*.*
3. *If* C *is a mmsc of* $\mathcal{I}$ *with role-depth* $\leq \delta$*, then* $\pi_{\mathcal{M}_{\mathcal{I},\delta}}(C)$ *is an intent of* $\mathbb{K}_{\mathcal{I},\delta}$*.*

Consequently, the mapping $\bigsqcap \colon \mathcal{M}_{\mathcal{I},\delta} \to \mathcal{ALEQR}^{\mathsf{Self}}(N_C, N_R)$ is an isomorphism from the intent-lattice $(\mathsf{Int}(\mathbb{K}_{\mathcal{I},\delta}), \cap)$ to the mmsc-lattice $(\mathsf{Mmsc}(\mathcal{I},\delta), \sqcup)$, and has the inverse $\pi_{\mathcal{M}_{\mathcal{I},\delta}}$. This shows the strong correspondence between the formal concept lattice of $\mathbb{K}_{\mathcal{I},\delta}$ and the description concept lattice of $\mathcal{I}$ w.r.t. role depth $\leq \delta$. We can infer the following corollary from Lemmata 12 and 19.

Corollary 20. *The intent lattice of* $\mathbb{K}_{\mathcal{I},\delta}$ *is isomorphic to the mmsc lattice of* $\mathcal{I}, \delta$*.*

We can further observe that the concept inclusions holding in $\mathcal{I}$ and the implications holding in $\mathbb{K}_{\mathcal{I},\delta}$ are also in a strong correspondence. We can show that whenever the implication $U \to V$ holds in $\mathbb{K}_{\mathcal{I},\delta}$, then also the concept inclusion $\bigsqcap U \sqsubseteq \bigsqcap V$ holds in $\mathcal{I}$. Furthermore, since every mmsc of $\mathcal{I}$ with a role depth $\leq \delta$ is expressible in terms of $\mathcal{M}_{\mathcal{I},\delta}$, and conjunctions of intents of $\mathbb{K}_{\mathcal{I},\delta}$ are exactly the mmscs of $\mathcal{I}$, and every concept inclusion $C \sqsubseteq D$ holding in $\mathcal{I}$ is entailed by the concept inclusion $C \sqsubseteq C^{\mathcal{I}\mathcal{I}}$, we can deduce that indeed every concept inclusion holding in $\mathcal{I}$ is entailed by the transformation of the canonical implicational base of $\mathbb{K}_{\mathcal{I},\delta}$, which consists of all GCIs that have a conjunction of a pseudo-intent as premise and the conjunction of the closure of the pseudo-intent as conclusion.

Lemma 21. *Let* $\mathcal{I}$ *be an interpretation over the signature* (N_C, N_R)*,* $\delta \in \mathbb{N}$ *a role-depth bound, and* $C \sqsubseteq D$ *a concept inclusion, such that both concepts* C, D *have a role-depth* $\leq \delta$*.*

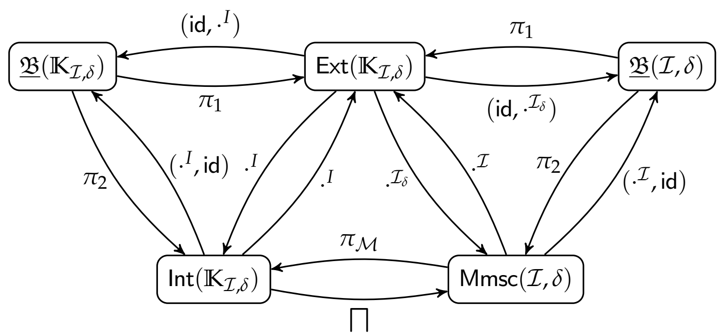

Fig. 4. Overview on the isomorphisms between the extent lattice, intent lattice, and mmsc lattice of $\mathbb{K}_{\mathcal{I},\delta}$ and $\mathcal{I},\delta$, respectively. Note that $\mathsf{Ext}(\mathbb{K}_{\mathcal{I},\delta}) = \mathsf{Ext}(\mathcal{I},\delta)$ holds.

1. *If D is expressible in terms of $\mathcal{M}_{\mathcal{I},\delta}$, and the implication $\pi_{\mathcal{M}_{\mathcal{I},\delta}}(C) \to \pi_{\mathcal{M}_{\mathcal{I},\delta}}(D)$ holds in $\mathbb{K}_{\mathcal{I},\delta}$, then the concept inclusion $C \sqsubseteq D$ holds in $\mathcal{I}$.*
2. *If C is expressible in terms of $\mathcal{M}_{\mathcal{I},\delta}$, and the concept inclusion $C \sqsubseteq D$ holds in $\mathcal{I}$, then the implication $\pi_{\mathcal{M}_{\mathcal{I},\delta}}(C) \to \pi_{\mathcal{M}_{\mathcal{I},\delta}}(D)$ holds in $\mathbb{K}_{\mathcal{I},\delta}$.*

Corollary 22 (Concept Inclusion Base). *Let $\mathcal{I}$ be an interpretation over the signature (N_C, N_R), and $\delta \in \mathbb{N}$ a role-depth bound. Then the following statements hold:*

1. *For all subsets $X, Y \subseteq \mathcal{M}_{\mathcal{I},\delta}$, the implication $X \to Y$ holds in $\mathbb{K}_{\mathcal{I},\delta}$ if, and only if, the concept inclusion $\bigsqcap X \sqsubseteq \bigsqcap Y$ holds in $\mathcal{I}$.*
2. *The intents of $\mathbb{K}_{\mathcal{I},\delta}$ are exactly the model-based most-specific concept descriptions of $\mathcal{I}$ with role-depth bound $\leq \delta$.*
3. *If $\mathcal{L}$ is an implicational base for $\mathbb{K}_{\mathcal{I},\delta}$, then $\bigsqcap \mathcal{L} := \{ \bigsqcap X \sqsubseteq \bigsqcap Y \mid X \to Y \in \mathcal{L} \}$ is a sound and complete TBox for all concept inclusions holding in $\mathcal{I}, \delta$. Especially this holds for the following TBox.*

$$\left\{ \bigsqcap P \sqsubseteq \bigsqcap P^{II} \,\middle|\, P \text{ is a pseudo-intent of } \mathbb{K}_{\mathcal{I},\delta} \right\}$$

6 Induced Role Contexts

Role contexts have been introduced by Zickwolff [21], and have been used by Rudolph [19] for gaining knowledge on binary relations or roles (that are interpreted as binary relations). We use their definition here for the deduction of complex role inclusions holding in an interpretation.

Definition 23 (Induced Role Context). *Let $\mathcal{I}$ be an interpretation over the signature (N_C, N_R), and $\delta \in \mathbb{N}$ a role depth bound. Furthermore, assume that $X = \{ x_0, x_1, \ldots, x_\delta \}$ is a set of $\delta + 1$* variables. *Then the* induced role context *for $\mathcal{I}$ and δ is defined as*

$$\mathbb{K}^R_{\mathcal{I},\delta} := \left(\left(\Delta^{\mathcal{I}} \right)^X, X \times N_R \times X, J \right)$$

where $(f, (x, r, y)) \in J$ if, and only if, $(f(x), f(y)) \in r^{\mathcal{I}}$.

Lemma 24 (Role Inclusions and Implications). *Let $\mathcal{I}$ be an interpretation over (N_C, N_R), $\delta \in \mathbb{N}$ a role-depth bound, and $n \leq \delta$. Then the complex role inclusion $r_1 \circ r_2 \circ \ldots \circ r_n \sqsubseteq s$ holds in $\mathcal{I}$ if, and only if, the implication $\{ (x_0, r_1, x_1), (x_1, r_2, x_2), \ldots, (x_{n-1}, r_n, x_n) \} \to \{ (x_0, s, x_n) \}$ holds in the induced role context $\mathbb{K}^R_{\mathcal{I},\delta}$.*

In particular, we are only interested in implications whose premise contains a subset of the form $\{ (x_0, r_1, x_1), (x_1, r_2, x_2), \ldots, (x_{k-1}, r_k, x_k) \}$. Hence, we define a constraining closure operator ϕ_R on the attribute set $X \times N_R \times X$ of the induced role context as follows.

$$\phi_R(B) := \begin{cases} B & \text{if } \exists k \in \mathbb{N}_+ \exists r_1, r_2, \ldots, r_k \in N_R \\ & \quad \exists \{ x_0, x_1, \ldots, x_k \} \in \binom{X}{k+1} : \\ & \quad \{ (x_0, r_1, x_1), \ldots, (x_{k-1}, r_k, x_k) \} \subseteq B, \\ B \cup \{ (x_0, r, x_1) \mid r \in N_R \} & \text{otherwise.} \end{cases}$$

We shall now formulate a base of all complex role inclusions holding in an interpretation. For this purpose, we refer to [15] for the notions of constrained implications and their bases. A *ϕ-constrained* implication over M is an implication $X \to Y$ over M such that both premise X and conclusion Y are ϕ-closed. A *ϕ-constrained* implicational base for a formal context $\mathbb{K}$ is a set of ϕ-constrained implications that is valid in $\mathbb{K}$, and furthermore entails all ϕ-constrained implications that hold in $\mathbb{K}$.

Theorem 25 (Role Inclusion Base). *Let $\mathcal{I}$ be an interpretation over (N_C, N_R). If $\mathcal{L}$ is a ϕ_R-constrained implicational base of $\mathbb{K}_{\mathcal{I},R}$, then the following RBox $\mathcal{R}_{\mathcal{I},\delta}$ is sound, complete, and irredundant, for all complex role inclusions holding in $\mathcal{I}, \delta$.*

$$\mathcal{R}_{\mathcal{I},\delta} := \left\{ r_1 \circ r_2 \circ \ldots \circ r_k \sqsubseteq s \;\middle|\; \begin{array}{l} \exists X \to Y \in \mathcal{L} \\ \exists r_1, r_2, \ldots, r_k, s \in N_R \\ \exists \{ x_0, x_1, \ldots, x_k \} \in \binom{X}{k+1} : \\ X \supseteq \{ (x_0, r_1, x_1), (x_1, r_2, x_2), \ldots, (x_{k-1}, r_k, x_k) \} \\ Y \ni (x_0, s, x_k) \end{array} \right\}$$

7 Construction of the Knowledge Base

By means of the results of the previous Sections 5 and 6 we are now ready to formulate a knowledge base for an interpretation $\mathcal{I}$, or for a description graph $\mathcal{G}$, respectively. Beforehand, it is necessary to inspect the interplay of role and concept inclusions to ensure irredundancy of the knowledge base. First, we list some trivial concept inclusions that hold in all interpretations.

Lemma 26. *Let $m, n \in \mathbb{N}_+$ be non-negative integers with $n < m$, $r \in N_R$ a role name, and C a concept description. The following general concept inclusions hold in every interpretation $\mathcal{I}$.*

$$
\begin{aligned}
A \sqcap \neg A &\sqsubseteq \bot \\
\exists r.\,\mathsf{Self} \sqcap \forall r.\,C &\sqsubseteq C \\
\exists r.\,\mathsf{Self} \sqcap C &\sqsubseteq \exists r.\,C \\
\exists r.\,\mathsf{Self} \sqcap C \sqcap \leq 1.\,r.\,C &\sqsubseteq \forall r.\,C \\
\exists r.\,C \sqcap \forall r.\,D &\sqsubseteq \exists r.\,(C \sqcap D) \\
\geq n.\,r.\,C \sqcap \forall r.\,D &\sqsubseteq \geq n.\,r.\,(C \sqcap D) \\
\leq n.\,r.\,C \sqcap \forall r.\,D &\sqsubseteq \leq n.\,r.\,(C \sqcap D) \\
\exists r.\,C &\sqsubseteq \geq 1.\,r.\,C \\
\geq n.\,r.\,C &\sqsubseteq \exists r.\,C \\
\leq n.\,r.\,C &\sqsubseteq \leq m.\,r.\,C \\
\geq m.\,r.\,C &\sqsubseteq \geq n.\,r.\,C \\
\geq \left|\Delta^{\mathcal{I}}\right|.\,r.\,C &\sqsubseteq C \sqcap \forall r.\,C \sqcap \exists r.\,\mathsf{Self} \\
\top &\sqsubseteq \leq \left|\Delta^{\mathcal{I}}\right|.\,r.\,C
\end{aligned}
$$

Please note that there are no direct subsumptions between existential restrictions $\exists r.\,C$ and value restrictions $\forall r.\,C$, i.e., both $\exists r.\,C \sqsubseteq \forall r.\,C$ and $\forall r.\,C \sqsubseteq \exists r.\,C$ do not hold. There is also a crossover between both constructors existential restriction and value restriction. The constructor is denoted by $\forall\!\exists$, and has the semantics $(\forall\!\exists\, r.\,C)^{\mathcal{I}} := (\exists r.\,C)^{\mathcal{I}} \cap (\forall r.\,C)^{\mathcal{I}}$, i.e., a domain element is in the extension of $\forall\!\exists\, r.\,C$ if, and only if, there is an r-successor in C, and all r-successors are in C.

The next two lemmata show us which concept inclusions can be inferred from known role inclusions.

Lemma 27. *Let $\mathcal{I}$ be a model of the role inclusion axiom $r \sqsubseteq s$, C an arbitrary concept description, $Q_1 \in \{\exists, \geq n\}$, $Q_2 \in \{\forall, \leq n\}$, and $n \in \mathbb{N}_+$. Then $\mathcal{I}$ is also a model of the following general concept inclusions.*

$$
\begin{aligned}
Q_1\, r.\,C &\sqsubseteq Q_1\, s.\,C \\
\exists r.\,\mathsf{Self} &\sqsubseteq \exists s.\,\mathsf{Self} \\
Q_2\, s.\,C &\sqsubseteq Q_2\, r.\,C
\end{aligned}
$$

Lemma 28. *Let $\mathcal{I}$ be a model of the complex role inclusion $r_1 \circ r_2 \circ \ldots \circ r_k \sqsubseteq s$, C an arbitrary concept description, $Q_1 \in \{\exists, \geq n\}$, $Q_2 \in \{\forall, \leq n\}$, and $n \in \mathbb{N}_+$. Then $\mathcal{I}$ is also a model of the following concept inclusions.*

$$
\begin{aligned}
\exists r_1.\,\exists r_2.\ldots Q_1\, r_k.\,C &\sqsubseteq Q_1\, s.\,C \\
Q_2\, s.\,C &\sqsubseteq \forall r_1.\,\forall r_2.\ldots Q_2\, r_k.\,C
\end{aligned}
$$

As final step we use the trivial concept inclusions and concept inclusions that are entailed by valid role inclusions to define some background knowledge for the computation of the canonical implicational base of the induced concept context which is trivial in terms of description logics, but not for formal concept analysis due to their different semantics.

Theorem 29 (Knowledge Base). *Let $\mathcal{I}$ be an interpretation over the signature (N_C, N_R), and $\delta \in \mathbb{N}$ a role-depth bound. Furthermore, assume that $\mathcal{L}$ is an implicational base of the induced concept context $\mathbb{K}^{C}_{\mathcal{I},\delta}$ w.r.t. the background knowledge*

$$\begin{aligned}
\mathcal{S}_{\mathcal{I}} := &\left\{ \{\, C \,\} \to \{\, D \,\} \,\middle|\, \begin{array}{l} C, D \in \mathcal{M}_{\mathcal{I},\delta}, \\ C \sqsubseteq D \end{array} \right\} \\
&\cup \{\, \{\, A, \neg A \,\} \to \mathcal{M}_{\mathcal{I},\delta} \mid A \in N_C \,\} \\
&\cup \left\{ \begin{array}{c} \left\{ \exists r.\, X^{\mathcal{I}_{\delta-1}}, \forall r.\, Y^{\mathcal{I}_{\delta-1}} \right\} \to \left\{ \exists r.\, Z^{\mathcal{I}_{\delta-1}} \right\}, \\ \left\{ \geq n.r.\, X^{\mathcal{I}_{\delta-1}}, \forall r.\, Y^{\mathcal{I}_{\delta-1}} \right\} \to \left\{ \geq n.r.\, Z^{\mathcal{I}_{\delta-1}} \right\}, \\ \left\{ \leq m.r.\, X^{\mathcal{I}_{\delta-1}}, \forall r.\, Y^{\mathcal{I}_{\delta-1}} \right\} \to \left\{ \leq m.r.\, Z^{\mathcal{I}_{\delta-1}} \right\}, \end{array} \middle| \begin{array}{l} r \in N_R, \\ \emptyset \neq X, Y, Z \subseteq \Delta^{\mathcal{I}}, \\ Z^{\mathcal{I}_{\delta-1}} \equiv X^{\mathcal{I}_{\delta-1}} \sqcap Y^{\mathcal{I}_{\delta-1}}, \\ 1 \leq m < n \leq \left|\Delta^{\mathcal{I}}\right| \end{array} \right\} \\
&\cup \left\{ \begin{array}{c} \left\{ \exists r.\, X^{\mathcal{I}_{\delta-1}} \right\} \to \left\{ \exists s.\, X^{\mathcal{I}_{\delta-1}} \right\}, \\ \left\{ \forall s.\, X^{\mathcal{I}_{\delta-1}} \right\} \to \left\{ \forall r.\, X^{\mathcal{I}_{\delta-1}} \right\}, \\ \left\{ \geq n.r.\, X^{\mathcal{I}_{\delta-1}} \right\} \to \left\{ \geq n.s.\, X^{\mathcal{I}_{\delta-1}} \right\}, \\ \left\{ \leq m.s.\, X^{\mathcal{I}_{\delta-1}} \right\} \to \left\{ \leq m.r.\, X^{\mathcal{I}_{\delta-1}} \right\}, \\ \{\, \exists r.\, \mathsf{Self} \,\} \to \{\, \exists s.\, \mathsf{Self} \,\} \end{array} \middle| \begin{array}{l} r, s \in N_R, \\ r \sqsubseteq s \in \mathcal{R}, \\ 1 \leq m < n \leq \left|\Delta^{\mathcal{I}}\right|, \\ \emptyset \neq X \subseteq \Delta^{\mathcal{I}} \end{array} \right\}.
\end{aligned}$$

Then $\mathcal{K}_{\mathcal{I},\delta} = (\mathcal{T}_{\mathcal{I},\delta}, \mathcal{R}_{\mathcal{I},\delta})$ is a knowledge base for $\mathcal{I}$ where $\mathcal{T}_{\mathcal{I},\delta} := \bigsqcap \mathcal{L}$ holds as in Corollary 22, and $\mathcal{R}_{\mathcal{I},\delta}$ is defined as in Theorem 25.

8 Other Description Logics

If only a lower expressivity of the underlying description logic is necessary, then one could also use $\mathcal{EL}$, $\mathcal{FLE}$, or extensions thereof with role hierarchies $\mathcal{H}$, or complex role inclusions $\mathcal{R}$. All of the previous results are still valid, however one has to remove some of the used concept descriptions that are not expressible in the chosen description logic. Figure 5 gives an overview on description logics that have a lower expressivity than $\mathcal{ALEQR}^{\mathsf{Self}}$, and could also be used for knowledge acquisition.

8.1 Role Hierarchies $\mathcal{H}$ instead of Complex Role Inclusions $\mathcal{R}$

In the special case of simple role inclusions provided by the extension $\mathcal{H}$ it is not necessary to use the induced role context. We can directly extract the role hierarchy from the interpretation $\mathcal{I}$, or the description graph $\mathcal{G}$, respectively, as follows.

First, we want to extract a minimal RBox $\mathcal{R}_{\mathcal{I}}$ from the interpretation that entails all role inclusion axioms holding in $\mathcal{I}$. We therefore define an equivalence relation $\equiv_{\mathcal{I}}$ on the role names as follows: $r \equiv_{\mathcal{I}} s$ if, and only if, $r^{\mathcal{I}} = s^{\mathcal{I}}$. Then let $N_R^{\mathcal{I}}$ be a set of representatives of this equivalence relation, i.e., $\left| N_R^{\mathcal{I}} \cap [r]_{\equiv_{\mathcal{I}}} \right| = 1$ for all role names $r \in N_R$. Then add the following role equivalence axioms to $\mathcal{R}_{\mathcal{I}}$: For each

constructor	$\mathcal{EL}$	$\mathcal{FL}_0$	$\mathcal{FLE}$	$\mathcal{ALE}$	$\mathcal{Q}$	Self	$\mathcal{H}$	$\mathcal{R}$
$\bot$				×				
$\top$	×	×	×	×				
$\neg A$				×				
$C \sqcap D$	×	×	×	×				
$\exists r. C$	×		×	×	×			
$\forall r. C$		×	×	×				
$\geq n. r. C$					×			
$\leq n. r. C$					×			
$\exists r. \mathsf{Self}$						×		
$C \sqsubseteq D$	×	×	×	×				
$C \equiv D$	×	×	×	×				
$r \sqsubseteq s$							×	×
$r_1 \circ \ldots \circ r_n \sqsubseteq s$								×

Fig. 5. Overview on various Description Logics below $\mathcal{ALEQR}^{\mathsf{Self}}$

representative role $r \in N_R^{\mathcal{I}}$, add the axioms $r \equiv s$ for all $s \in [r]_{\equiv_{\mathcal{I}}} \setminus \{r\}$. Furthermore, define an order relation $\sqsubseteq_{\mathcal{I}}$ on the representatives $N_R^{\mathcal{I}}$ by $r \sqsubseteq_{\mathcal{I}} s$ if, and only if, $r^{\mathcal{I}} \subseteq s^{\mathcal{I}}$. Let $\prec_{\mathcal{I}}$ be the neighborhood relation of $\sqsubseteq_{\mathcal{I}}$, then add the role inclusion axioms $r \sqsubseteq s$ for each pair $r \prec_{\mathcal{I}} s$ to the RBox $\mathcal{R}_{\mathcal{I}}$. Obviously, the constructed RBox is minimal w.r.t. the property to entail all valid role inclusion axioms holding in the interpretation $\mathcal{I}$. Eventually, the RBox in $\mathcal{K}_{\mathcal{I}}$ is defined as follows.

$$\mathcal{R}_{\mathcal{I}} := \{\, r \equiv s \mid r \in N_R^{\mathcal{I}}, s \in [r]_{\equiv_{\mathcal{I}}} \setminus \{\, r \,\} \,\} \cup \{\, r \sqsubseteq s \mid r, s \in N_R^{\mathcal{I}}, r \prec_{\mathcal{I}} s \,\}$$

9 Conclusion

We have provided an extension of the results of Baader and Distel [1, 2, 9] for the deduction of knowledge bases from interpretations in the more expressive description logic $\mathcal{ALEQR}^{\mathsf{Self}}$ w.r.t. descriptive semantics and role-depth bounds. Since role-depth-bounded model-based most-specific concept descriptions always exist, this technique can always be applied. Furthermore, the construction of knowledge bases has been reduced to the computation of implicational bases of formal contexts, which is a well-understood problem that has several available algorithms – for example the standard *NextClosure* algorithm from Ganter [10], or the parallel algorithm that has been introduced in [15] and implemented in [14]. The presented methods are prototypically implemented in *Concept Explorer FX* [14].

References

[1] Franz Baader and Felix Distel. “A Finite Basis for the Set of $\mathcal{EL}$-Implications Holding in a Finite Model”. In: *Formal Concept Analysis, 6th International Conference, ICFCA 2008, Montreal, Canada, February 25-28, 2008, Proceedings*. Ed. by

Raoul Medina and Sergei A. Obiedkov. Vol. 4933. Lecture Notes in Computer Science. Springer, 2008, pp. 46–61.

[2] Franz Baader and Felix Distel. "Exploring Finite Models in the Description Logic $\mathcal{EL}_{\text{gfp}}$". In: *Formal Concept Analysis, 7th International Conference, ICFCA 2009, Darmstadt, Germany, May 21-24, 2009, Proceedings*. Ed. by Sébastien Ferré and Sebastian Rudolph. Vol. 5548. Lecture Notes in Computer Science. Springer, 2009, pp. 146–161.

[3] Franz Baader and Bariş Sertkaya. "Applying Formal Concept Analysis to Description Logics". In: *Concept Lattices, Second International Conference on Formal Concept Analysis, ICFCA 2004, Sydney, Australia, February 23-26, 2004, Proceedings*. Ed. by Peter W. Eklund. Vol. 2961. Lecture Notes in Computer Science. Springer, 2004, pp. 261–286.

[4] Franz Baader et al. *Completing Description Logic Knowledge Bases using Formal Concept Analysis*. LTCS-Report 06-02. Dresden, Germany: Chair for Automata Theory, Institute for Theoretical Computer Science, Technische Universität Dresden, 2006.

[5] Franz Baader et al., eds. *The Description Logic Handbook: Theory, Implementation, and Applications*. New York, NY, USA: Cambridge University Press, 2003.

[6] Daniel Borchmann. "Learning Terminological Knowledge with High Confidence from Erroneous Data". PhD thesis. Dresden, Germany: Technische Universität Dresden, 2014.

[7] Daniel Borchmann. "Towards an Error-Tolerant Construction of $\mathcal{EL}^{\bot}$-Ontologies from Data Using Formal Concept Analysis". In: *Formal Concept Analysis, 11th International Conference, ICFCA 2013, Dresden, Germany, May 21-24, 2013. Proceedings*. Ed. by Peggy Cellier, Felix Distel, and Bernhard Ganter. Vol. 7880. Lecture Notes in Computer Science. Springer, 2013, pp. 60–75.

[8] Daniel Borchmann and Felix Distel. "Mining of $\mathcal{EL}$-GCIs". In: *Data Mining Workshops (ICDMW), 2011 IEEE 11th International Conference on, Vancouver, BC, Canada, December 11, 2011*. Ed. by Myra Spiliopoulou et al. IEEE Computer Society, 2011, pp. 1083–1090.

[9] Felix Distel. "Learning Description Logic Knowledge Bases from Data using Methods from Formal Concept Analysis". PhD thesis. Dresden, Germany: Technische Universität Dresden, 2011.

[10] Bernhard Ganter. "Two Basic Algorithms in Concept Analysis". In: *Formal Concept Analysis, 8th International Conference, ICFCA 2010, Agadir, Morocco, March 15-18, 2010. Proceedings*. Ed. by Léonard Kwuida and Baris Sertkaya. Vol. 5986. Lecture Notes in Computer Science. Springer, 2010, pp. 312–340.

[11] Bernhard Ganter and Sergei O. Kuznetsov. "Pattern Structures and Their Projections". In: *Conceptual Structures: Broadening the Base, 9th International Conference on Conceptual Structures, ICCS 2001, Stanford, CA, USA, July 30-August 3, 2001, Proceedings*. Ed. by Harry S. Delugach and Gerd Stumme. Vol. 2120. Lecture Notes in Computer Science. Springer, 2001, pp. 129–142.

[12] Bernhard Ganter and Rudolf Wille. *Formal Concept Analysis - Mathematical Foundations*. Springer, 1999.

[13] Jean-Luc Guigues and Vincent Duquenne. "Familles minimales d'implications informatives résultant d'un tableau de données binaires". In: *Mathématiques et Sciences Humaines* 95 (1986), pp. 5–18.

[14] Francesco Kriegel. *Concept Explorer FX*. Software for Formal Concept Analysis. 2010-2015. URL: `https://github.com/francesco-kriegel/conexp-fx`.

[15] Francesco Kriegel. *Next Closures – Parallel Exploration of Constrained Closure Operators*. LTCS-Report 15-01. Dresden, Germany: Chair for Automata Theory, Institute for Theoretical Computer Science, Technische Universität Dresden, 2015.

[16] Michael Luxenburger. “Implications partielles dans un contexte”. In: *Mathématiques, Informatique et Sciences Humaines* 29.113 (1991), pp. 35–55.

[17] Michael Luxenburger. “Implikationen, Abhängigkeiten und Galois-Abbildungen”. PhD thesis. TH Darmstadt, 1993.

[18] Michael Luxenburger. “Partielle Implikationen und partielle Abhängigkeiten zwischen Merkmalen”. Diploma Thesis. TH Darmstadt, 1988.

[19] Sebastian Rudolph. “Relational Exploration - Combining Description Logics and Formal Concept Analysis for Knowledge Specification”. PhD thesis. Dresden, Germany: Technische Universität Dresden, 2006.

[20] Bariş Sertkaya. “Formal Concept Analysis Methods for Description Logics”. PhD thesis. Dresden, Germany: Technische Universität Dresden, 2007.

[21] Monika Zickwolff. “Rule Exploration: First Order Logic in Formal Concept Analysis”. PhD thesis. Germany: Technische Hochschule Darmstadt, 1991.