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Research article

A second order numerical method for a Volterra integro-differential equation with a weakly singular kernel

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Abstract: In this paper, a second finite difference method on a graded grid is proposed for a Volterra integro-differential equation with a weakly singular kernel. The proposed scheme is obtained by using the two-step backward differentiation formula (BDF2) to discretize the first derivative term and the first-order interpolation scheme to approximate the integral term. The analysis of stability is proved and used to prove the convergence of our presented numerical method in the discrete maximum norm. Finally, Numerical experiments are given to verify the theoretical results.

Keywords: Volterra integro-differential equation; weakly singular kernel; orthogonal convolution kernels; graded mesh

1. Introduction

This article aims to study a second-order numerical method for the following Volterra integro-differential equation (VIDE) with a weakly singular kernel

$$\begin{cases} \mathcal{L}u := u'(x) + a(x)u(x) + \displaystyle\int_0^x (x-t)^{-\alpha} b(t)u(t)dt = f(x), \ x \in \Omega := (0, L], \\ u(0) = \mu, \end{cases} \tag{1.1}$$

where $\alpha \in (0,1)$. $a(x)$, $b(x)$, and $f(x)$ are smooth functions, and μ is an initial data. We assume that there exist two positive constants β_1, β_2 such that

$$|a(x)| \le \beta_1, \quad |b(x)| \le \beta_2, \quad x \in [0, L]. \tag{1.2}$$

To simplify the presentation, we assume that β_1, $\beta_2 \le \beta^*$, where β^* is a positive constant. Based on the above assumptions, the problem (1.1) has a unique solution, $u(x)$, which satisfies the following lemma:

Lemma 1.1. *[1, Theorem 4.1] If f can be written $f(x) = f_1(x) + x^\beta f_2(x)$, where $\beta > 0$ and $\beta \neq 1, 2, \ldots, N$. Then there exist positive constants C, d such that*

$$\left|u^k(x)\right| \leq Cd^k\Gamma(k+1)x^{\delta-k},\ x > 0,\ k = 1, 2, \ldots, N, \tag{1.3}$$

where $\delta = \min(2-\alpha, 1+\beta)$.

It is well known that Volterra integro-differential equations widely exist in biology, finance, population growth models and other fields (see, e.g., [2,3]). In recent years, there has been tremendous interest in developing finite difference [4–6], finite element [7, 8], and spectral methods [9–11] for first-order and second-order Volterra integro-differential equations. Among the existing numerical methods, most of the researchers focus their attention on high-accuracy finite element methods and spectral methods. Therefore, it is very necessary to study a class of high-order finite difference methods for VIDE(s).

As far as we know, linear multi-step methods with a uniform time grid are widely used in discretizing the first-order time derivative of partial differential equations (see [12, 13], for example). It should be pointed out that the variable step-size linear multi-step methods allow us to take different step-sizes for different scales, i.e., small step-sizes for the domain with solution rapidly varying and large for the domain with solution slowly changing. Therefore, the variable step-size linear multi-step methods demonstrate the prominent advantages of high accuracy compared to the constant step-size linear multi-step methods. Recently, Liao and Zhang [14] developed the variable two-step backward differentiation formula (BDF2) to discretize the time derivative of diffusion equations and gave a new theoretical framework by using the positive semi-definiteness of BDF2 convolution kernels and a class of orthogonal convolution kernels for the first time. Furthermore, Liao et al. [15] derived the stability and convergence analysis of the second-order backward difference formula (BDF2) with variable steps for the molecular beam epitaxial model without slope selection. Wang et al. [16] gave stability and error estimates for time discretizations of linear and semi-linear parabolic equations by the two-step backward differentiation formula (BDF2) method with variable step sizes. In [17], the authors proposed linearly implicit backward differentiation formulas with variable step sizes to solve the two-dimensional incompressible Navier Stokes equations.

Inspired by the above references [14–17], the main purpose of this paper is to develop a second-order finite difference scheme for VIDE (1.1). This paper is organized as follows: In Section 2, we proposes a finite difference scheme on a graded mesh by using the variable step-size BDF2 to discretize the first derivative term and the linear interpolation technique to approximate the integral term and prove the stability of our proposed numerical method. In Section 3, we will show how to improve the point-wise accuracy to second order by selecting appropriate graded mesh parameters. Finally, the theoretical results are verified by numerical experiments in Section 4.

2. Discretization scheme

Let $\bar{\Omega}^N := \{0 = x_0 < x_1 < \cdots < x_N = L\}$ be a graded mesh (see [18]), where the grid points are given by $x_i = L\left(\frac{i}{N}\right)^\gamma$, $i = 0, 1, \cdots, N$ and $\gamma \in [1, \infty)$ is a given real number. For $i = 1, 2, \ldots, N$, let $h_i = x_i - x_{i-1}$ be the $i-$th mesh step and $h = \max\limits_{1\leq i\leq N} h_i$ be the corresponding maximum step size. Furthermore, we denote the $i-$th step-size ratios by $r_i = \frac{h_i}{h_{i-1}}$, $i = 2, \ldots, N$. Obviously, $r_{\max} = \max\limits_{2\leq i\leq N-1} r_i = 2^\gamma - 1$.

Throughout this paper, for any continuous function $g(x)$, let $g_i = g(x_i)$, and C represents a positive constant independent of the mesh parameter N.

Let $P_{i,k}g$ be the Lagrange interpolating polynomial of a function g over points $x_i, x_{i-1}, \ldots, x_{i-k}$. Then the BDF2 formula can be given by

$$D_2 g_i := (P_{i,2}g)'(x_i) = \frac{1+2r_i}{h_i(1+r_i)}\nabla g_i - \frac{r_i^2}{h_i(1+r_i)}\nabla g_{i-1} \ \text{ for } \ i \geq 2, \tag{2.1}$$

where $\nabla g_i := g_i - g_{i-1}$. In addition, denote $D_2 g_1 := \nabla g_1/h_1$. Then, on the above graded mesh $\bar{\Omega}^N$, we construct the following discretization scheme to approximate problem (1.1):

$$\begin{cases} \mathcal{L}^N u_i^N := D_2 u_i^N + a_i u_i^N + \displaystyle\sum_{k=1}^{i} \int_{x_{k-1}}^{x_k} (x_i - s)^{-\alpha} \left(\overline{bu}\right)(s) ds = f_i, \\ u_0^N = \mu, \end{cases} \tag{2.2}$$

where u_i^N is the approximation solution of $u(x)$ at point $x = x_i$ and

$$\left(\overline{bu}\right)(x) := b_k u_k^N + (x - x_k)\frac{b_k u_k^N - b_{k-1}u_{k-1}^N}{h_k}, \quad x \in (x_{k-1}, x_k).$$

In the following numerical analysis, the method of the discrete orthogonal convolution (DOC) kernels θ_{i-j}^i will play an important role. Firstly, we rewrite the BDF2 formula (2.1) into the following formal

$$D_2 g_i = \sum_{k=1}^{i} b_{i-k}^i \nabla g_k \ \text{ for } \ i \geq 1, \tag{2.3}$$

where b_{i-k}^i are defined by $b_0^1 := 1/h_1$, and

$$b_0^i := \frac{1+2r_i}{h_i(1+r_n)}, \ b_1^i := -\frac{r_i^2}{h_i(1+r_i)} \ \text{ and } \ b_j^i := 0 \ \text{ for } \ 2 \leq j \leq i-1. \tag{2.4}$$

The DOC kernels θ_{i-j}^i have the following the property (see [14])

$$\sum_{j=k}^{i} \theta_{i-j}^i b_{j-k}^j = \delta_{ik} \ \text{ for } \ \forall \ 1 \leq k \leq i,$$

where δ_{ik} is the Kronecker delta symbol. Furthermore, we have

$$\sum_{j=1}^{i} \theta_{i-j}^i D_2 u_j = \sum_{l=1}^{i} \nabla u_i \sum_{j=l}^{i} \theta_{i-j}^i b_{j-l}^j = u_i - u_{i-1}, \quad 1 \leq i \leq N. \tag{2.5}$$

We now introduce the following lemma for the DOC kernels θ_{i-j}^i.

Lemma 2.1. *[14, Lemma 2.3] For $i \geq 1$, the DOC kernels θ_{i-j}^i satisfy the following formula*

$$\theta_{i-j}^i > 0, \ \forall \ 1 \geq j \geq i \ \textit{ and } \ \sum_{j=1}^{i} \theta_{i-j}^i = h_i. \tag{2.6}$$

Next, the stability result of the discretization scheme (2.2) is given as follows:

Lemma 2.2. *Assume that* $h \le (1-\alpha)/(16\beta^*)$. *Then the discrete solution* u_i^N *of the discretization scheme (2.2) satisfies the following formula*

$$\left|u_n^N\right| \le C\left(\left|u_0^N\right| + \sum_{i=1}^{n}\sum_{j=1}^{i}\theta_{i-j}^{i}\left|f_j\right|\right),\ n \ge 1.$$

Proof. Firstly, for the integral term of (2.2), we have

$$\begin{aligned}
&\left|\sum_{k=1}^{j}\int_{x_{k-1}}^{x_k}\left(x_j-s\right)^{-\alpha}\left(\overline{bu}\right)(s)ds\right| \\
\le&\left|\sum_{k=1}^{j}b_k u_k^N\int_{x_{k-1}}^{x_k}\left(x_j-s\right)^{-\alpha}ds\right| + \left|\sum_{k=1}^{j}\frac{\nabla b_k u_k^N}{h_k}\int_{x_{k-1}}^{x_k}\left(x_j-s\right)^{-\alpha}(s-x_k)\,ds\right| \\
\le&\,\beta^*\max_{1\le k\le j}\left|u_k^N\right|\int_0^{x_j}\left(x_j-s\right)^{-\alpha}ds + \sum_{k=1}^{j}\left|\nabla b_k u_k^N\right|\int_{x_{k-1}}^{x_k}\left(x_j-s\right)^{-\alpha}ds \\
\le&\,\frac{\beta^*}{1-\alpha}\max_{1\le k\le j}\left|u_k^N\right| + 2\beta^*\max_{1\le k\le j}\left|u_k^N\right|\int_0^{x_j}\left(x_j-s\right)^{-\alpha}ds \\
\le&\,\frac{3\beta^*}{1-\alpha}\max_{1\le k\le j}\left|u_k^N\right|.
\end{aligned} \tag{2.7}$$

For $i \ge 1$, multiplying both sides of the discretization scheme (2.2) by the DOC kernels θ_{i-j}^{i} and summing j from 1 to i yields,

$$\sum_{j=1}^{i}\theta_{i-j}^{i}\left(D_2 u_j^N + a_j u_j^N + \sum_{k=1}^{j}\int_{x_{k-1}}^{x_k}\left(x_j-s\right)^{-\alpha}\left(\overline{bu}\right)(s)ds\right) = \sum_{j=1}^{i}\theta_{i-j}^{i}f_j. \tag{2.8}$$

Applying Eq (2.5) to Eq (2.8), one has

$$u_i^N - u_{i-1}^N + \sum_{j=1}^{i}\theta_{i-j}^{i}a_j u_j^N + \sum_{j=1}^{i}\theta_{i-j}^{i}\sum_{k=1}^{j}\int_{x_{k-1}}^{x_k}\left(x_j-s\right)^{-\alpha}\left(\overline{bu}\right)(s)ds = \sum_{j=1}^{i}\theta_{i-j}^{i}f_j. \tag{2.9}$$

Multiplying both sides of the above equation (2.9) by $2u_i^N$ and summing the resulting equality from 1 to n, one has

$$\begin{aligned}
\sum_{i=1}^{n}(u_i^N - u_{i-1}^N)^2 + \left|u_n^N\right|^2 - \left|u_0^N\right|^2 =& \sum_{i=1}^{n}2u_i^N\sum_{j=1}^{i}\theta_{i-j}^{i}f_j - \sum_{i=1}^{n}2u_i^N\sum_{j=1}^{i}\theta_{i-j}^{i}a_j u_j^N \\
&- \sum_{i=1}^{n}2u_i^N\sum_{j=1}^{i}\theta_{i-j}^{i}\sum_{k=1}^{j}\int_{x_{k-1}}^{x_k}\left(x_j-s\right)^{-\alpha}\left(\overline{bu}\right)(s)ds.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}\left|u_n^N\right|^2 &\le \left|u_0^N\right|^2 + \sum_{i=1}^{n}\left|2u_i^N\right|\sum_{j=1}^{i}\theta_{i-j}^{i}\left|f_j\right| + \beta^*\sum_{i=1}^{n}\left|2u_i^N\right|\sum_{j=1}^{i}\theta_{i-j}^{i}\left|u_j^N\right| \\ &\quad + \sum_{i=1}^{n}\left|2u_i^N\right|\sum_{j=1}^{i}\theta_{i-j}^{i}\left|\sum_{k=1}^{j}\int_{x_{k-1}}^{x_k}\left(x_j - s\right)^{-\alpha}\left(\overline{bu}\right)(s)ds\right| \\ &\le \left|u_0^N\right|^2 + \sum_{i=1}^{n}\left|2u_i^N\right|\sum_{j=1}^{i}\theta_{i-j}^{i}\left|f_j\right| + \beta^*\sum_{i=1}^{n}\left|2u_i^N\right|\sum_{j=1}^{i}\theta_{i-j}^{i}\left|u_j^N\right| \\ &\quad + \frac{3\beta^*}{1-\alpha}\sum_{i=1}^{n}\left|2u_i^N\right|\sum_{j=1}^{i}\theta_{i-j}^{i}\left|\max_{1\le k\le j}u_k^N\right|.\end{aligned} \tag{2.10}$$

Set $\left|u_m^N\right| := \max_{0\le i\le n}\left|u_i^N\right|$, letting $n = m$ in the inequality (2.10), we get

$$\begin{aligned}\left|u_m^N\right|^2 &\le \left|u_0^N\right|\left|u_m^N\right| + \left|2u_m^N\right|\sum_{i=1}^{m}\sum_{j=1}^{i}\theta_{i-j}^{i}\left|f_j\right| + \beta^*\left|u_m^N\right|\sum_{i=1}^{m}\left|2u_i^N\right|\sum_{j=1}^{i}\theta_{i-j}^{i} \\ &\quad + \frac{3\beta^*}{1-\alpha}\left|u_m^N\right|\sum_{i=1}^{m}\left|2u_i^N\right|\sum_{j=1}^{i}\theta_{i-j}^{i}.\end{aligned} \tag{2.11}$$

Using Lemma 2.1 to Eq (2.11), one has

$$\begin{aligned}\left|u_n^N\right| \le \left|u_m^N\right| &\le \left|u_0^N\right| + 2\sum_{i=1}^{m}\sum_{j=1}^{i}\theta_{i-j}^{i}\left|f_j\right| + 2\beta^*\sum_{i=1}^{m}\left|u_i^N\right|h_i + \frac{6\beta^*}{1-\alpha}\sum_{i=1}^{m}\left|u_i^N\right|h_i \\ &\le \left|u_0^N\right| + 2\sum_{i=1}^{n}\sum_{j=1}^{i}\theta_{i-j}^{i}\left|f_j\right| + \left(\frac{8\beta^*}{1-\alpha}\right)\sum_{i=1}^{n}\left|u_i^N\right|h_i.\end{aligned} \tag{2.12}$$

If $h \le (1-\alpha)/(16\beta^*)$, we have

$$\left|u_n^N\right| \le 2\left|u_0^N\right| + 4\sum_{i=1}^{n}\sum_{j=1}^{i}\theta_{i-j}^{i}\left|f_j\right| + \left(\frac{16\beta^*}{1-\alpha}\right)\sum_{i=1}^{n-1}\left|u_i^N\right|h_i. \tag{2.13}$$

Based on the discrete Grönwall inequality [19, Lemma 3.2], we have

$$\begin{aligned}\left|u_n^N\right| &\le \left(2\left|u_0^N\right| + 4\sum_{i=1}^{n}\sum_{j=1}^{i}\theta_{i-j}^{i}\left|f_j\right|\right)\exp\left(1 + \left(\frac{16\beta^*}{1-\alpha}\right)\sum_{i=1}^{n-1}h_i\right) \\ &\le C\left(\left|u_0^N\right| + \sum_{i=1}^{n}\sum_{j=1}^{i}\theta_{i-j}^{i}\left|f_j\right|\right),\end{aligned}$$

which completes the proof. □

3. Error analysis

For $i = 0, 1, \cdots, N$, let $e_i = u_i - u_i^N$ be the absolution error between numerical solution u_i^N and exact solution $u(x)$ at point $x = x_i$. Then the error equation can be written by

$$\mathcal{L}^N \left(u_i - u_i^N\right) = R_{1,i} + R_{2,i},\ i = 1, 2, 3, \ldots, N, \tag{3.1}$$

where $R_{1,i},\ R_{2,i}$ are characterized by

$$R_{1,i} = D_2 u_i - u'(x_i), \tag{3.2}$$

$$R_{2,i} = \sum_{k=1}^{i} \int_{x_{k-1}}^{x_k} (x_i - s)^{-\alpha} \left[\left(\overline{bu}\right)(s) - (bu)(s)\right] ds. \tag{3.3}$$

Lemma 3.1. *The truncation error $R_{1,i}$ and $R_{2,i}$ estimations can be given by the following inequality:*

$$\begin{aligned} &\left|R_{1,i}\right| \le CN^{\gamma-\gamma\delta},\ i = 1, 2, \\ &\left|R_{1,i}\right| \le C\left(h_i^2 x_{i-1}^{\delta-3} + h_{i-1}^2 x_{i-2}^{\delta-3}\right),\ \ i \ge 3, \\ &\left|R_{2,i}\right| \le C \max\left\{N^{-\gamma\delta}, N^{-2}\right\},\ \ i \ge 1. \end{aligned}$$

Proof. For the truncation error $R_{1,1}$, it follows from Taylor's expansion formula and Lemma 1.1 that

$$\left|R_{1,1}\right| = \left|\frac{1}{h_1}\int_0^{x_1} t u''(t)dt\right| \le \int_0^{x_1} t^{\delta-2} dt \le C h_1^{\delta-1} \le CN^{\gamma-\gamma\delta}. \tag{3.4}$$

Similarly, for $i \ge 2$, based on [14], one has

$$\begin{aligned} \left|R_{1,i}\right| = &\left|\frac{1+2r_i}{2(1+r_i)h_i}\int_{x_{i-1}}^{x_i} (t - x_{i-1})^2 u'''(t)dt - \frac{r_i^2}{2h_i(1+r_i)}\int_{x_{i-2}}^{x_{i-1}} (t - x_{i-2})^2 u'''(t)dt\right. \\ &\left. - \frac{r_i}{2(1+r_i)}\int_{x_{i-1}}^{x_i} (2(t - x_{i-1}) + h_{i-1}) u'''(t)dt\right|. \end{aligned} \tag{3.5}$$

Furthermore, based on Eq (3.5), yields,

$$\begin{aligned} \left|R_{1,2}\right| &\le \frac{1+2r_2}{2(1+r_2)} h_2^2 x_1^{\delta-3} + \frac{r_2}{2h_1(1+r_2)}\int_0^{x_1} t^{\delta-1} dt + \frac{r_2+1}{2(1+r_2)} h_2^2 x_1^{\delta-3} \\ &\le C h_2^2 h_1^{\delta-3} + C h_1^{\delta-1} \le C h_1^{\delta-1}\left(r_2^2 + 1\right) \le C h_1^{\delta-1} \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \left|R_{1,i}\right| &\le \frac{1+2r_i}{2(1+r_i)} h_i^2 x_{i-1}^{\delta-3} + \frac{r_i}{2(1+r_i)} h_{i-1}^2 x_{i-2}^{\delta-3} + \frac{r_i+1}{2(1+r_i)} h_i^2 x_{i-1}^{\delta-3} \\ &\le C\left(h_i^2 x_{i-1}^{\delta-3} + h_{i-1}^2 x_{i-2}^{\delta-3}\right),\ i \ge 3. \end{aligned} \tag{3.7}$$

For $R_{2,i}$, $1 \le i \le N$, it is easy to obtain

$$\begin{aligned}
\left|R_{2,i}\right| &= \left|\sum_{k=1}^{i}\int_{x_{k-1}}^{x_k}(x_i-s)^{-\alpha}\left[\frac{x_k-s}{h_k}\int_{x_{k-1}}^{x_k}(t-x_{k-1})\,u''(t)dt+\int_{s}^{x_k}(t-s)\,u''(t)dt\right]ds\right| \\
&\le \sum_{k=1}^{i}\int_{x_{k-1}}^{x_k}(x_i-s)^{-\alpha}\left[\int_{x_{k-1}}^{x_k}(t-x_{k-1})\left|u''(t)\right|dt+\int_{s}^{x_k}(t-s)\left|u''(t)\right|dt\right]ds \\
&\le \sum_{k=1}^{i}\int_{x_{k-1}}^{x_k}(x_i-s)^{-\alpha}\,2\int_{x_{k-1}}^{x_k}(t-x_{k-1})\left|u''(t)\right|dtds \qquad (3.8)\\
&\le 2\max_{1\le k\le i}\int_{x_{k-1}}^{x_k}(t-x_{k-1})\left|u''(t)\right|dt\sum_{k=1}^{i}\int_{x_{k-1}}^{x_k}(x_i-s)^{-\alpha}\,ds \\
&\le C\max_{1\le k\le i}\left(\int_{x_{k-1}}^{x_k}t^{\delta/2-1}dt\right)^2 \le C\max_{1\le k\le i}\left(x_k^{\delta/2}-x_{k-1}^{\delta/2}\right)^2,
\end{aligned}$$

where we have used the following inequality:

$$\int_a^b \phi(s)(s-a)ds \le \frac{1}{2}\left[\int_a^b \sqrt{\phi(s)}ds\right]^2,$$

for any decreasing function $\phi > 0$ on $[a, b]$, see [20]. When graded mesh parameter $\gamma \le 2/\delta$, Eq (3.8), the following estimates can be given:

$$\left|R_{2,i}\right| \le C\max_{1\le k\le i}\left[\left(\frac{k}{N}\right)^{\gamma\delta/2}-\left(\frac{k-1}{N}\right)^{\gamma\delta/2}\right]^2 \le CN^{-\gamma\delta}, \qquad (3.9)$$

where the following inequality is used

$$a^p - b^p \le (a-b)^p, \;\; 0 < p < 1, \; 0 \le b \le a.$$

Conversely, if $\gamma > 2/\delta$,

$$\begin{aligned}
\left|R_{2,i}\right| &\le C\max_{1\le k\le i}\left[\left(\frac{k}{N}\right)^{\gamma\delta/2}-\left(\frac{k-1}{N}\right)^{\gamma\delta/2}\right]^2 \\
&\le C\max_{1\le k\le i}\left(\frac{\gamma\delta}{2N}\xi_k^{\gamma\delta/2-1}\right)^2 \le CN^{-2},
\end{aligned} \qquad (3.10)$$

where $\xi_k \in \left(\frac{k-1}{N}, \frac{k}{N}\right)$. To sum up, this lemma is proved. □

Now, we derive the main result of this paper as follows:

Theorem 3.1. *Let u_n be the exact solution of problem (1.1) at the point $x = x_n$ and u_n^N be the solution of problem (2.2) on the mesh $\bar{\Omega}^N$. Then, under the condition $h \le (1-\alpha)/(16\beta^*)$, we have*

$$\max_{0\le n\le N}\left|u_n - u_n^N\right| \le \begin{cases} CN^{-\gamma\delta}, & \gamma < 2/\delta, \\ CN^{-2}\log N, & \gamma = 2/\delta, \\ CN^{-2}, & \gamma > 2/\delta. \end{cases} \qquad (3.11)$$

Proof. For $n \geq 1$, Lemma 2.2 and Eq (3.1) imply that

$$\left|u_n - u_n^N\right| \leq C \sum_{i=1}^{n} \sum_{j=1}^{i} \theta_{i-j}^{i} \left|R_{1,j} + R_{2,j}\right| \leq C \left(P_n + Q_n\right), \tag{3.12}$$

where

$$P_n = \sum_{i=1}^{n} \sum_{j=1}^{i} \theta_{i-j}^{i} \left|R_{1,j}\right|,$$
$$Q_n = \sum_{i=1}^{n} \sum_{j=1}^{i} \theta_{i-j}^{i} \left|R_{2,j}\right|.$$

Based on Lemma 2.2 and Lemma 3.1, it is easy to obtain

$$Q_n \leq C \max\left\{N^{-\gamma\delta}, N^{-2}\right\} \sum_{i=1}^{n} \sum_{j=1}^{i} \theta_{i-j}^{i} \leq C \max\left\{N^{-\gamma\delta}, N^{-2}\right\},\ n \geq 1. \tag{3.13}$$

Next, for P_n, the following estimate can be obtained by exchanging the summation order

$$P_n \leq \sum_{j=1}^{n} \left|R_{1,j}\right| \sum_{i=j}^{n} \theta_{i-j}^{i} \leq \sum_{j=1}^{n} \left|R_{1,j}\right| h_j \leq \sum_{j=1}^{2} \left|R_{1,j}\right| h_j + \sum_{j=3}^{n} \left|R_{1,j}\right| h_j, \tag{3.14}$$

where the fact $\sum\limits_{i=j}^{n} \theta_{i-j}^{i} \leq C h_j$ is used. Then it is easy to get the following estimations:

$$\begin{aligned} \sum_{j=1}^{2} \left|R_{1,j}\right| h_j &\leq C N^{\gamma-\gamma\delta} \sum_{j=1}^{2} h_j \leq C N^{\gamma-\gamma\delta} x_2 \\ &\leq C N^{\gamma-\gamma\delta} (2/N)^{\gamma} \leq C N^{-\gamma\delta}, \end{aligned} \tag{3.15}$$

and

$$\sum_{j=3}^{n} \left|R_{1,j}\right| h_j \leq C \sum_{j=3}^{n} \left(h_j^3 x_{j-1}^{\delta-3} + h_j h_{j-1}^2 x_{j-2}^{\delta-3}\right). \tag{3.16}$$

Furthermore, by using $h_j \leq T\gamma N^{-1} (j/N)^{\gamma-1}$, yields,

$$\begin{aligned} \sum_{j=3}^{n} \left|R_{1,j}\right| h_j &\leq C N^{-3} \sum_{j=3}^{n} \left[\left(\frac{j}{N}\right)^{3(\gamma-1)} \left(\frac{j-1}{N}\right)^{\gamma(\delta-3)} + \left(\frac{j}{N}\right)^{3(\gamma-1)} \left(\frac{j-2}{N}\right)^{\gamma(\delta-3)}\right] \\ &\leq C \sum_{j=3}^{n} \frac{j^{3(\gamma-1)}}{N^{\gamma\delta}} \left[(j-1)^{\gamma\delta-3\gamma} + (j-2)^{\gamma\delta-3\gamma}\right] \frac{j^{\gamma\delta-3\gamma}}{j^{\gamma\delta-3\gamma}} \\ &\leq C \sum_{j=3}^{n} \frac{j^{\gamma\delta-3}}{N^{\gamma\delta}} \left[(1-1/j)^{\gamma\delta-3\gamma} + (1-2/j)^{\gamma\delta-3\gamma}\right] \\ &\leq C \sum_{j=3}^{n} \frac{j^{\gamma\delta-3}}{N^{\gamma\delta}} \left[(2/3)^{\gamma\delta-3\gamma} + (1/3)^{\gamma\delta-3\gamma}\right] \leq C \sum_{j=3}^{n} \frac{j^{\gamma\delta-3}}{N^{\gamma\delta}}. \end{aligned} \tag{3.17}$$

If $\gamma < 2/\delta$, we have

$$\sum_{j=3}^{n} \frac{j^{\gamma\delta-3}}{N^{\gamma\delta}} \leq CN^{-\gamma\delta}. \tag{3.18}$$

If $\gamma = 2/\delta$, by using $\sum\limits_{j=3}^{n} j^{-1} \leq \int_1^n \frac{1}{x} dx \leq \ln N$, yields,

$$\sum_{j=3}^{n} \frac{j^{\gamma\delta-3}}{N^{\gamma\delta}} \leq CN^{-2} \int_3^n j^{-1} dt \leq CN^{-2} \ln N. \tag{3.19}$$

If $\gamma > 2/\delta$, the following estimation will be obtained through the definition of the definite integral

$$\sum_{j=3}^{n} \frac{j^{\gamma\delta-3}}{N^{\gamma\delta}} = N^{-2} \sum_{j=3}^{n} \frac{1}{N} \left(\frac{j}{N}\right)^{\gamma\delta-3} \leq N^{-2} \int_0^1 x^{\gamma\delta-3} dx \leq CN^{-2}. \tag{3.20}$$

According to Eqs (3.14)–(3.20), the estimation of P_n in Eq (3.12) can be obtained. The desirable result can be followed by Eq (3.13).

□

4. Numerical results and discussion

In order to verify our theoretical results, we consider the following test problem [1]

$$u'(x) + u(x) + \int_0^x (x-t)^{-\alpha} e^t u(t) dt = f(x), \ x \in \Omega := (0, L], \tag{4.1}$$

$$u(0) = \mu, \tag{4.2}$$

where $f = (2-\alpha) x^{1-\alpha} + \frac{\Gamma(1-\alpha)\Gamma(3-\alpha)}{\Gamma(4-2\alpha)} x^{3-2\alpha}$. The exact solution of this problem is $u(x) = x^{2-\alpha} e^{-x}$. Since the exact solution is given, the maximum absolute error and the convergence order are calculated as follows:

$$E^N := \max_{0 \leq i \leq N} \left| u_i^N - u_i \right|, \ \ \rho = \log_2 \left(\frac{E^N}{E^{2N}} \right).$$

For different mesh parameters γ, N and α, Table 1 shows the maximum absolute errors and convergence orders. It is shown from these numerical experiments that they complement the theoretical results given in Theorem 3.1.

Table 1. The maximum errors and convergence orders

α	N	$\gamma = 1$		$\gamma = 2$		$\gamma = 3$	
		E^N	ρ	E^N	ρ	E^N	ρ
0.1	2^9	8.7481e-06	1.88	2.0699e-06	2.00	3.5185e-06	2.00
	2^{10}	2.3645e-06	1.89	5.1713e-07	2.00	8.7881e-07	2.00
	2^{11}	6.3633e-07	1.89	1.2924e-07	2.00	2.1960e-07	2.00
	2^{12}	1.7087e-07	1.89	3.2304e-08	2.00	5.4887e-08	2.00
	2^{13}	4.5833e-08	-	8.0754e-09	-	1.3720e-08	-
	$\gamma\delta$		1.90		3.8		5.7
0.3	2^9	2.1574e-05	1.68	1.9564e-06	2.00	2.8607e-06	2.00
	2^{10}	6.6875e-06	1.69	4.8865e-07	2.00	7.1450e-07	2.00
	2^{11}	2.0656e-06	1.69	1.2210e-07	2.00	1.7854e-07	2.00
	2^{12}	6.3687e-07	1.69	3.0518e-08	2.00	4.4626e-08	2.00
	2^{13}	1.9619e-07	-	7.6284e-09	-	1.1155e-08	-
	$\gamma\delta$		1.70		3.40		5.10
0.5	2^9	4.9960e-05	1.48	1.9545e-06	2.00	2.2527e-06	2.00
	2^{10}	1.7808e-05	1.49	4.8840e-07	2.00	5.6267e-07	2.00
	2^{11}	6.3214e-06	1.49	1.2205e-07	2.00	1.4062e-07	2.00
	2^{12}	2.2394e-06	1.49	3.0503e-08	2.00	3.5152e-08	2.00
	2^{13}	7.9251e-07	-	7.6243e-09	-	8.7881e-09	-
	$\gamma\delta$		1.50		3.00		4.50
0.7	2^9	9.9815e-05	1.28	2.0758e-06	1.98	1.5693e-06	2.00
	2^{10}	4.0911e-05	1.29	5.2525e-07	1.99	3.9187e-07	2.00
	2^{11}	1.6689e-05	1.29	1.3231e-07	1.99	9.7955e-08	2.00
	2^{12}	6.7922e-06	1.29	3.3236e-08	1.99	2.4496e-08	2.00
	2^{13}	2.7614e-06	-	8.3348e-09	-	6.1268e-09	-
	$\gamma\delta$		1.30		2.60		3.90
0.9	2^9	1.0708e-04	1.05	1.6942e-06	1.86	5.9577e-07	2.00
	2^{10}	5.1652e-05	1.07	4.6535e-07	1.89	1.4805e-07	2.00
	2^{11}	2.4483e-05	1.08	1.2550e-07	1.91	3.6917e-08	2.00
	2^{12}	1.1510e-05	1.09	3.3382e-08	1.92	9.2226e-09	2.00
	2^{13}	5.3894e-06	-	8.7848e-09	-	2.3062e-09	-
	$\gamma\delta$		1.10		2.20		3.30

5. Concluding remarks

Based on the variable step size BDF2, this paper proposes a second-order numerical method on the graded mesh to solve a Volterra integro-differential equation with a weak singular kernel and gives rigorous stability and convergence analysis for our presented numerical method. In the further work, by using the analysis of BDF3 given in [21], we shall study a third-order numerical method for the Volterra integro-differential equation with a weakly singular kernel.

Author contributions

Li-Bin Liu: Methodolog, writing review and editing; Limin Ye: software and writing original draft preparation; Xiaobing Bao: software, formal analysis and writing review and editing; Yong Zhang: numerical analysis and check the English writing. Further, all the authors have checked and approved the final version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of Interest

The authors declare there is no conflict of interest.

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